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THE STEADY MOTION OF A CIRCULAR CRACK ALONG THE INTERFACE OF A LAYER AND A HALF-SPACE†

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An exact analytical solution is given of the direct axisymmetric steady dynamic problem in the theory of elasticity of the motion, with an arbitrary constant subsonic velocity $\upsilon = c$ along the interface of a rigidly coupled layer $0 \le z \le H$ and a half-space $z \le 0$, of a circular transverse shear crack $0 \le r \le r^0 + ct$ ($r^0 \le 0$) with and without a cavity at its tip. Using Hankel transformation in terms of biwave potentials, a general solution of the fundamental equations of motion in the theory of elasticity and the basic solutions of the first fundamental boundary-value problem are separately constructed for the layer and the half-space for the case of arbitrary normal and shear stresses in the plane of separation $z = 0$ in a moving cylindrical system of coordinates $r_1 = r + ct, z_1 = z$. A special regularization of the main solution is carried out which ensures the convergence of the integrals for all stresses and displacements while preserving the high accuracy of the solution to whatever level may be desired [1, 2]. On the basis of the main solutions, a mathematical formulation is given of the mixed problem of the motion of a transverse shear crack with a cavity at the tip and its reduction to a system of three singular integral equations with Cauchy kernels which allows of regularization by the Carleman–Vekua method in terms of the closed solution of the corresponding characteristic system of singular integral equations. When the width of the cavity vanishes, one of the equations of the system solves the problem of a transverse shear crack without a cavity. Criteria are established for the existence of a cavity and its absence as a function of the elastic and velocity characteristics of the layer and half-space and the velocity of motion of the crack c.

The problem described is intended for investigating interference waves as well as proper surface and boundary Rayleigh and Stoneley waves which are generated by a moving crack according to the law of synchronism. Its possible relation to the source of moving forces of an earthquake can be perceived.

1. FORMULATION OF THE BASIC AND MIXED PROBLEMS

The two-layer half-space which is considered consists of a layer of arbitrary thickness H and a foundation layer of infinite thickness (a homogeneous half space) to which the numbers 1 and 2 are assigned respectively. Young's moduli of elasticity E_i, Poisson's ratios ν_i and the density ρ_i of the material ($i = 1, 2$) can take different and arbitrary values. We will take the origin of a cylindrical system of coordinates r, z in the plane of separation of the layers and we will direct the Oz axis upwards, orthogonal to the layers. In this system of coordinates, the upper layer $0 \le z \le H$ and the foundation layer $z \le 0$ are separated by the plane $z = 0$, and the plane $z = H$ is the upper boundary of the upper layer (Fig. 1).

The outer boundary $z = H$ is stress free. At the instant of time $t = 0$ in the plane of separation of the layers $z = 0$, a circular transverse shear crack $0 \le r + ct \le r^0 + ct$ ($r^0 \ge 0$) occurs spontaneously and starts to move at an arbitrary constant subsonic velocity $\upsilon = c$ and, outside the crack, the conditions for the rigid coupling of the layers, which ensures the continuity of the normal and shear components of the stresses and displacements, must be satisfied. We will initially assume that the edges of the crack are in contact everywhere, have bilateral bonds and, during the motion, rub against one another with friction, the law of which is subject to determination from the condition of the synthesis of all of the boundary conditions in the plane of the crack $0 \le r + ct, z = 0$. This formulation of the problem will

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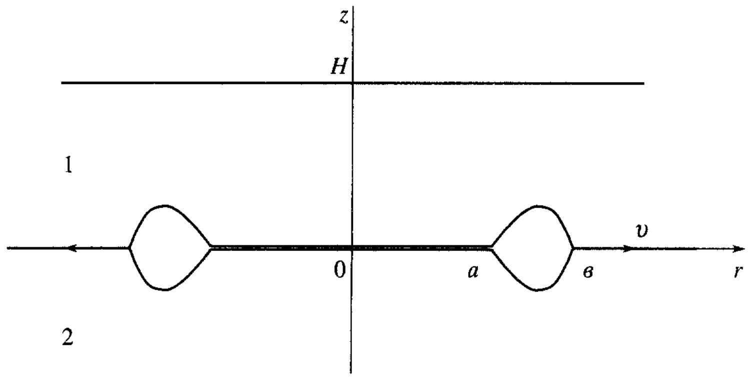

Fig. 1

only be correct when the axial normal stresses in the whole of the crack domain $0 \leq r + ct \leq r^0 + ct$ are compressive stresses (of negative sign). However, if a zone (zones) of tensile normal stresses appears in the crack domain, then, when there are unilateral bonds in this zone (zones), the edges of the crack will tear away from one another and from a cavity (cavities). It is only possible to establish the signs of the normal axial stresses in the whole crack domain by a numerical analysis of the problem. The crack tip $r = r^0 + ct$, at which the normal axial stresses undergo an infinite discontinuity, is an exception. The analytical solution of the initial problem of a crack without a cavity enables us to prove that the normal axial stress intensity factor at the crack tip changes sign over the range of change in the elastic and velocity characteristics of the layers at any subsonic velocity of the crack propagation. Consequently, in the case of certain elastic and velocity characteristics of the layers, a zone of normal tensile stresses appears in the neighbourhood of the tip and therefore, a cavity is formed at the tip under conditions when there is unilateral bonding of the edges of the crack in this zone.

The above conditions give a sufficient basis for formulating the key problems being considered of a transverse shear crack (without a cavity) and with a single cavity at the tip. A numerical realization of the analytical solutions of these problems will enable us to establish the specific characteristics of the layers and the crack propagation velocity $\upsilon = c$ for which they are correct and to investigate the possibility of the appearance of intermediate cavities in the case of other characteristics of the problem. This, in turn, will provide sufficient grounds for formulating the generalized problem of the propagation of a transverse shear crack when there is an arbitrary finite number of cavities.

The analytical solutions of the fundamental and mixed problems described below are given in a moving cylindrical system of coordinates $r_1 = r + ct$, $z_1 = z$ in the dimensionless variables $\rho = r_1/b$, $\zeta = z_1/H$, where $b = r^0 + ct$ is the value of the radius of the circle, taken as the linear unit of measurement. Here, it should be recalled and explained that, in the steady-state problem being considered, all the mechanical characteristics and the required stresses and displacements are independent of the time t. One should therefore consider the ratio $\rho = r_1/b$ as an independent dimensionless variable in the radial semi-axis $0 \leq \rho$. The upper layer ($i = 1$) is located in the interval $0 \leq \zeta \leq 1$ in the dimensionless vertical semi-axis $O\zeta$ and the foundation layer ($i = 2$) is located in the unbounded interval $\zeta \leq 0$. The crack is located in the plane $\zeta = 0$ in the area of the circle $0 \leq \rho \leq 1$ and the cavity is located in the area of the ring $\alpha^0 \leq \rho \leq 1$, where $\alpha^0 = a/b$ is the dimensionless radius of the internal contour of the cavity which is to be determined from the condition that the normal axial stress intensity factor vanishes (Fig. 1). The magnitudes of the ratios

$$\lambda = H/b, \quad \delta = E_1/E_2, \quad \chi = \delta(1+\nu_2)/(1+\nu_1), \quad G_i = E_i/(2(1+\nu_i)), \quad i = 1, 2 \tag{1.1}$$

are the characteristic geometrical and elastic parameters of the mixed problem and the velocities of the longitudinal stress-strain waves c_{1i} and the transverse shear waves c_{2i}

$$c_{1i} = \sqrt{\frac{2(1+\nu_i)G_i}{1-2\nu_i}\frac{}{\rho_i}}, \quad c_{2i} = \sqrt{\frac{G_i}{\rho_i}}, \quad i = 1, 2 \tag{1.2}$$

are the velocity characteristics of the layers. We will denote the normal axial and shear stresses and the axial and tangential displacements in a layer with number $i = 1, 2$, by $\sigma_{zi}(\rho, \zeta)$, $\tau_{rzi}(\rho, \zeta)$, $w_i(\rho, \zeta)$ and $u_i(\rho, \zeta)$ respectively.

2. THE GENERAL SOLUTION OF THE FUNDAMENTAL EQUATIONS OF MOTION

The fundamental equations of motion in the theory of elasticity in the variables $u_i(r, z)$ and $w_i(r, z)$ in the layers $i = 1, 2$

$$\left(\Delta - \frac{1}{r^2}\right)u_i + \frac{1}{1-2\nu_i}\frac{\partial l_i}{\partial r} = \frac{\rho_i}{G_i}\frac{\partial^2 u_i}{\partial t^2}, \quad \Delta w_i + \frac{1}{1-2\nu_i}\frac{\partial l_i}{\partial z} = \frac{\rho_i}{G_i}\frac{\partial^2 w_i}{\partial t^2}$$

$$\Delta = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{\partial^2}{\partial z^2}, \quad l_i = \frac{\partial u_i}{\partial r} + \frac{u_i}{r} + \frac{\partial w_i}{\partial z}$$

in the moving system of coordinates $r_1 = r + ct$, $z_1 = z$ can be written, taking into account the relations

$$\frac{\partial^2 u_i}{\partial t^2} = c^2\left(\frac{1}{r_1}\frac{\partial}{\partial r_1}\left(r_1\frac{\partial}{\partial r_1}\right) - \frac{1}{r_1^2}\right)u_i, \quad \frac{\partial^2 w_i}{\partial t^2} = c^2\frac{1}{r_1}\frac{\partial}{\partial r_1}\left(r_i\frac{\partial}{\partial r_1}\right)w_i$$

in the form

$$\left(\Delta_{2i} - \frac{k_{2i}^2}{r_1^2}\right)u_i + \frac{1}{1-2\nu_i}\frac{\partial l_i}{\partial r_1} = 0, \quad \Delta_{2i}w_i + \frac{1}{1-2\nu_i}\frac{\partial l_i}{\partial z_1} = 0 \tag{2.1}$$

$$\Delta_{2i} = k_{2i}^2\frac{1}{r_1}\frac{\partial}{\partial r_1}\left(r_1\frac{\partial}{\partial r_1}\right) + \frac{\partial^2}{\partial z_1^2}, \quad k_{2i} = \sqrt{1 - \frac{c^2}{c_{2i}^2}}, \quad l_i = \frac{\partial u_i}{\partial r_1} + \frac{u_i}{r_1} + \frac{\partial w_i}{\partial z_1} \tag{2.2}$$

The general solution of the equations is obtained in the form

$$u_i = -\frac{1+\nu_i}{E_i}\frac{\partial^2 \varphi_i}{\partial r_1 \partial z_1}, \quad w_i = \frac{1+\nu_i}{E_i}\left(2(1-\nu_i)\Delta_{2i} + \frac{c^2}{c_{2i}^2}\frac{1}{r_1}\frac{\partial}{\partial r_1}\left(r_1\frac{\partial}{\partial r_1}\right) - \frac{\partial^2}{\partial z_1^2}\right)\varphi_i \tag{2.3}$$

where $\varphi_i(r_1, z_1)$ are arbitrary functions which satisfy the biwave condition

$$\Delta_{1i}\Delta_{2i}\varphi_i(r_1, z_1) = 0 \tag{2.4}$$

with the operators Δ_{2i} (2.2) and

$$\Delta_{1i} = k_{1i}^2\frac{1}{r_1}\frac{\partial}{\partial r_1}\left(r_1\frac{\partial}{\partial r_1}\right) + \frac{\partial^2}{\partial z_1^2}, \quad k_{1i} = \sqrt{1 - \frac{c^2}{c_{1i}^2}} \tag{2.5}$$

The normal axial and shear stresses are determined in terms of the displacements using Hooke's law

$$\begin{aligned}\sigma_{zi} &= \frac{\partial}{\partial z_1}\left((2-\nu_i)\Delta_{2i} + \frac{c^2}{c_{2i}^2}\frac{1}{r_1}\frac{\partial}{\partial r_1}\left(r_1\frac{\partial}{\partial r_1}\right) - \frac{\partial^2}{\partial z_1^2}\right)\varphi_i(r_1, z_1)\\ \tau_{rzi} &= \frac{\partial}{\partial r_1}\left((1-\nu_i)\Delta_{2i} + \frac{1}{2}\frac{c^2}{c_{2i}^2}\frac{1}{r_1}\frac{\partial}{\partial r_1}\left(r_1\frac{\partial}{\partial r_1}\right) - \frac{\partial^2}{\partial z_1^2}\right)\varphi_i(r_1, z_1)\end{aligned} \tag{2.6}$$

In the case of a subsonic velocity of motion of the crack when the conditions $1 - c^2/c_{2i}^2 > 0$ $(i = 1, 2)$, for which the radicals k_{1i} (2.5) and k_{2i} (2.2) take real values, are satisfied, for the biwave functions $\varphi_i(r_1, z_1)$ $(i = 1, 2)$ we adopt the general solution of Eq. (2.4) in the form of the Hankel integral

$$\varphi_i = \int_0^\infty (A_i^*(\alpha)e^{k_{2i}\alpha z_1} + B_i^*(\alpha)e^{k_{1i}\alpha z_1} + C_i^*(\alpha)e^{-k_{2i}\alpha z_1} + D_i^*(\alpha)e^{-k_{1i}\alpha z_1})J_0(\alpha r_1)d\alpha \tag{2.7}$$

where A_i^*, B_i^*, C_i^* and D_i^* are arbitrary functions of the variable of integration, which are to be determined from the boundary conditions for the main and mixed problems with the exception of $C_2^*(\alpha) \equiv D_2^*(\alpha) \equiv 0$ $(0 \le \alpha < \infty)$. The functions $\varphi_i(r_1, z_1)$ in the form of the integral (2.7) are biwave potentials in terms of which, using formulae (2.3) and (2.6), we obtain constructive expressions for the displacements and stresses in the layers $i = 1, 2$. In these expressions, we introduce the dimensionless variables $\rho = r_1/b$, $\zeta = z_1/H$ and the new unknown functions $A_i(\beta)$, $B_i(\beta)$, $C_i(\beta)$, $D_i(\beta)$ of the parameter $\beta = b\alpha$ with the appropriate normalizing factors which eliminate increasing exponents and thereby ensure the correctness of the solution. As a result, we can represent the normal and shear stresses and the axial and radial displacements in the form

$$\sigma_{zi} = \int_0^\infty \beta\Delta_{zi}(\zeta, \beta)J_0(\rho\beta)d\beta, \quad \tau_{rzi} = \int_0^\infty \beta\Delta_{\tau i}(\zeta, \beta)J_1(\rho\beta)d\beta$$
$$\frac{E_i}{(1+\nu_i)b}w_i = \int_0^\infty \Delta_{wi}(\zeta, \beta)J_0(\rho\beta)d\beta, \quad \frac{E_i}{(1+\nu_i)b}u_i = \int_0^\infty \Delta_{ui}(\zeta, \beta)J_1(\rho\beta)d\beta \tag{2.8}$$

$$\begin{aligned}
\Delta_{zi} &= -A_i k_{2i}\eta_{2i} + B_i n_i \eta_{1i} + C_i k_{2i}\psi_{2i} - D_i n_i \psi_{1i}\\
\Delta_{\tau i} &= A_i m_i \eta_{2i} + B_i s_i \eta_{1i} + C_i m_i \psi_{2i} + D_i s_i \psi_{1i}\\
\Delta_{wi} &= -A_i \eta_{2i} + B_i f_i \eta_{1i} - C_i \psi_{2i} + D_i f_i \psi_{1i}\\
\Delta_{ui} &= A_i k_{2i}\eta_{2i} + B_i k_{1i}\eta_{1i} - C_i k_{2i}\psi_{2i} - D_i k_{1i}\psi_{1i}
\end{aligned} \tag{2.9}$$

$$\begin{aligned}
&\eta_{ji} = \exp(-\lambda\beta k_{ji}(\zeta_{i-1} - \zeta)), \quad \psi_{ji} = \exp(-\lambda\beta k_{ji}(\zeta - \zeta_i)), \quad \zeta_0 = 1, \quad \zeta_1 = 0\\
&n_i = k_{1i}((1-\nu_i)(k_{1i}^2 - k_{2i}^2) - 1), \quad m_i = (1 + k_{2i}^2)/2\\
&s_i = \nu_i(k_{1i}^2 - k_{2i}^2) + (1 + k_{2i}^2)/2, \quad f_i = (1 - 2\nu_i)(k_{1i}^2 - k_{2i}^2) - 1
\end{aligned} \tag{2.10}$$

3. REGULARIZED BASIC SOLUTION FOR THE UPPER LAYER

The regularized basic solution for the upper layer is constructed with the following boundary conditions on the outer surface $\zeta = 1$ and on the interface of the layers $\zeta = 0$

$$\begin{aligned}
\sigma_{z1} &= p^*(\rho), \quad \tau_{rz1} = q^*(\rho) \ \text{when} \ \zeta = 1\\
\sigma_{z1} &= p(\rho), \quad \tau_{rz1} = q(\rho) \ \text{when} \ \zeta = 0
\end{aligned} \tag{3.1}$$

where $p(\rho)$ and $q(\rho)$ are arbitrary functions on the semi-axis $0 \le \rho < \infty$, which can be represented by the Hankel integrals

$$p(\rho) = \int_0^\infty \beta\bar{p}(\beta)J_0(\rho\beta)d\beta, \quad q(\rho) = \int_0^\infty \beta\bar{q}(\beta)J_1(\rho\beta)d\beta \tag{3.2}$$

$$\bar{p}(\beta) = \int_0^\infty \rho p(\rho)J_0(\rho\beta)d\rho, \quad \bar{q}(\beta) = \int_0^\infty \rho q(\rho)J_1(\rho\beta)d\rho \tag{3.3}$$

$p^*(\rho)$ and $q^*(\rho)$ $(0 \le \rho < \infty)$ are functions which may be as small as desired, which are intended for regularizing the solution of problem (3.1)–(3.3). We will seek the regularized solution of this problem in the form of the superposition of the separate solutions corresponding to a normal load $p(\rho)$ with small overload $p^*(\rho)$ when $q(\rho) \equiv q^*(\rho) \equiv 0$ and a shear load $q(\rho)$ with a small overload $q^*(\rho)$ when $p(\rho) \equiv p^*(\rho) \equiv 0$. The small overload functions are represented in the form

$$p^*(\rho) = -\int_0^\infty \beta R(\beta)\Delta_{wp1}(1,\beta)J_0(\rho\beta)d\beta$$
$$q^*(\rho) = -\int_0^\infty \beta R(\beta)\Delta_{uq1}(1,\beta)J_1(\rho\beta)d\beta \tag{3.4}$$

where $\Delta_{wp1}(1, \beta)$, $\Delta_{uq1}(1, \beta)$ are functions of the compliance, which are represented by the formulae $\Delta_{w1}(\zeta, \beta)$, $\Delta_{u1}(\zeta, \beta)$ (2.9) on the outer surface of the layer $\zeta = 1$ in the case of the normal load $p(\rho)$ and the shear load $q(\rho)$ respectively. At the same time, we assume that

$$R(\beta) = \varepsilon\exp(-n\beta), \quad 0<\varepsilon \ll 1, \quad n \gg 1 \tag{3.5}$$

Note that the introduction of the overload functions $p^*(\rho)$ and $q^*(\rho)$ into the boundary conditions (3.1) is solely intended to ensure, via the function $R(\beta)$ (3.5), the convergence of the integrals (2.8) for the displacements which, in the case of $R(\beta) \equiv 0$ and, therefore, in the case of $p^*(\rho) \equiv q^*(\rho) \equiv 0$, diverge in the lower limits. However, at the same time, it is required that the moduli $|p^*(\rho)|$ and $|q^*(\rho)|$ $(0 \le \rho < \infty)$ and the modulus of the principal loading vector $p^*(\rho)$ (the principal overload vector $q^*(\rho)$ vanishes according to the symmetry condition) do not exceed the magnitude $\delta = \delta(\varepsilon, n) > 0$, which may be as small as desired and depends on the constants ε and n of the function $R(\beta)$ (3.5). When $\varepsilon \to 0$ and $n \to \infty$, we have $\delta(\varepsilon, n) = O(n^{-1}\sqrt{\varepsilon/n})$ and, therefore, the functions $p^*(\rho)$ and $q^*(\rho)$ in boundary conditions (3.1) can be interpreted as infinitesimal functions of the regularization of the basic solutions while preserving its form as high a degree of accuracy as desired.

Substituting formulae (2.8), (3.2) and (3.4) into boundary conditions (3.1), we obtain the boundary conditions for the functions $\Delta_{\upsilon s1}(\zeta, \beta)$ $(\upsilon = z, \tau, w, u)$ (2.9) expressed in terms of the sets of unknown functions $A_{s1}(\beta)$, $B_{s1}(\beta)$, $C_{s1}(\beta)$ and $D_{s1}(\beta)$ $(0 \le \rho < \infty)$ which have been given the subscript s in the case of a normal load $(s = p)$

$$\Delta_{zp1}(1,\beta) + R(\beta)\Delta_{wp1}(1,\beta) = 0, \quad \Delta_{\tau p1}(1,\beta) = 0$$
$$\Delta_{zp1}(0,\beta) = \bar{p}(\beta), \quad \Delta_{\tau p1}(0,\beta) = 0 \tag{3.6}$$

and in the case of a shear load $(s = q)$

$$\Delta_{zq1}(1,\beta) = 0, \quad \Delta_{\tau q1}(1,\beta) + R(\beta)\Delta_{uq1}(1,\beta) = 0$$
$$\Delta_{zq1}(0,\beta) = 0, \quad \Delta_{\tau q1}(0,\beta) = \bar{q}(\beta) \tag{3.7}$$

The equalities (3.6) and (3.7) in expanded form, with formulae (2.9) substituted into them, are the correct systems of functional equations (SFE) for the determining the unknown functions $A_{s1}(\beta)$, $B_{s1}(\beta)$, $C_{s1}(\beta)$ and $D_{s1}(\beta)$ $(s = p, q)$ in the case of normal and shear loads. On solving the expanded SFE (3.6) and (3.7) using Cramer's rule, we find these unknown functions, expressed in terms of the transforms $\bar{p}(\beta)$ and $\bar{q}(\beta)$ (3.3) respectively, in an analytical form and substitute them into the general solution (2.8). A a result, we obtain the regularized solutions of the boundary-value problem for the upper layer $(i = 1)$ separately for the normal load $p(\rho)$ and shear load $q(\rho)$ on its lower boundary plane $\zeta = 0$. The superposition of these solutions gives the required regularized basic solution of problem (3.1)–(3.3) for the upper layer $(i = 1)$.

The following representations of the axial and radial displacements $w_1(\rho, \zeta)$ and $u_1(\rho, \zeta)$ on the lower boundary of the layer $\zeta = 0$

$$w_1(\rho,0) = \int_0^\infty \Delta_{w1}(\beta)J_0(\rho\beta)d\beta, \quad u_1(\rho,0) = \int_0^\infty \Delta_{u1}(\beta)J_1(\rho\beta)d\beta \tag{3.8}$$

$$\Delta_{\upsilon 1} = E_1^{-1}(1+\nu_1)bD_{\upsilon 1}(\beta), \quad D_{\upsilon 1}(\beta) = \Delta_{\upsilon p1}(\beta)\bar{p}(\beta) + \Delta_{\upsilon q1}(\beta)\bar{q}(\beta); \quad \upsilon = w, u \tag{3.9}$$

are required to formulate the mixed problem of the crack propagation from the basic solution described above.

Omitting the unwieldy analytical expressions for the functions $\Delta_{\upsilon s1}(\beta)$ ($\upsilon = w, u; s = p, q$), which can be obtained using the method described above with the transforms $\bar{p}(\beta)$ and $\bar{q}(\beta)$ in the SFE (3.6) and (3.7) replaced by unity, we will present only the leading terms of their asymptotic expansions when $\beta \to 0$

$$\begin{aligned} \Delta_{wp1}(\beta), \Delta_{uq1}(\beta) &= -\varepsilon^{-1} + O(\beta) \\ (\Delta_{wq1}(\beta), \Delta_{up1}(\beta)) &= -2\left(\frac{1}{1-k_{21}^2}, \frac{\nu_1}{1+\nu_1-(1-\nu_1)k_{21}^2}\right) + O(\beta) \end{aligned} \tag{3.10}$$

and when $\beta \to \infty$

$$\begin{aligned} \left\| \begin{matrix} \Delta_{wp1}(\beta) & \Delta_{wq1}(\beta) \\ \Delta_{up1}(\beta) & \Delta_{uq1}(\beta) \end{matrix} \right\| &= \left\| \begin{matrix} -A_{wp1} & A_{wq1} \\ A_{up1} & -A_{uq1} \end{matrix} \right\| + O(\exp(-2\lambda k_{21}\beta)) \\ A_{\upsilon s1} &= \frac{a_{\upsilon s}}{R_{l1}}, \quad \upsilon = w, u; \quad s = p, q \end{aligned} \tag{3.11}$$

Here

$$\begin{aligned} a_{wp1} &= (k_{11}+k_{21})(1+(1-2\nu_1)k_{21}^2) \\ a_{up1} &= 1+k_{21}^2-2\nu_1 k_{21}(k_{11}+k_{21}) \\ a_{wq1} &= 2(1-(k_{11}+k_{21})((1-\nu_1)k_{11}-(1-2\nu_1)k_{21})) \\ a_{uq1} &= 2(1-\nu_1)k_{11}k_{21}(k_{11}+k_{21}) \\ R_{l1} &= 2\nu_1 k_{21}(k_{11}+k_{21})+(1+k_{21}^2)((1-\nu_1)k_{11}(k_{11}+k_{21})-1) \end{aligned} \tag{3.12}$$

On the basis of the asymptotic formulae (3.10)–(3.12), it is possible to give a rigorous proof of the convergence of the regularized integrals (3.8), regardless of the unknown bounded transforms $\bar{p}(\beta)$ and $\bar{q}(\beta)$, which enables us to reduce the initial mixed problem to a system of singular integral equations, which is developed below.

In concluding this section, it is necessary in passing to dwell on the properties of the asymptotic functions (3.11) and (3.12) at infinity, which determine the form of the solution of the mixed problem and the singularities at the crack tip as a function of its velocity $\upsilon = c > 0$. All the quantities $A_{\upsilon s1}$, $a_{\upsilon s1}$ ($\upsilon = w, u; s = p, q$), R_{l1} (3.10), (3.11) depend, via the radicals k_{21} (2.2) and k_{11} (2.5), on the constant subsonic velocity $c < c_{21}$ as a parameter of the actual problem and are therefore functions of c. In this case, the properties

$$a_{\upsilon s1}(c) > 0, \quad \upsilon = w, u; \quad s = p, q \tag{3.13}$$

are either obvious or are easily proved.

The function $R_{l1}(c)$ (3.12) plays a special role in the solution of the problem of the crack propagation. It is an analogue of Rayleigh's function in the similar dynamic problem of surface waves in a homogeneous half-space. The zeros of the function $R_{l1}(c)$ are determined from the irrational algebraic equation $R_{l1}(c) = 0$, which reduces to the following well-known rational algebraic Rayleigh equation in $x_1 = c^2/c_{21}^2$:

$$(1-\nu_1)x_1^3 - 8(1-\nu_1)x_1^2 + 8(2-\nu_1)x_1 - 8 = 0 \tag{3.14}$$

The resonance velocity $c_R = c_{21}\sqrt{x_{1R}}$ of the natural non-dispersing Rayleigh waves, which are moving along the outer surface of the two-layer half-space, are determined in terms of the real root of Eq. (3.14) x_{1R}.

4. THE BASIC SOLUTION FOR THE FOUNDATION LAYER

The basic solution of the first fundamental problem for the foundation layer ($i = 2$) must satisfy the same boundary conditions at the interface of the layers $\zeta = 0$ as in the analogous problem for the upper layer, namely, conditions (3.1)

$$\sigma_{z2} = p(\rho), \quad \tau_{rz2} = q(\rho) \quad \text{when} \quad \zeta = 0 \tag{4.1}$$

The functions $p(\rho)$ and $q(\rho)$ are defined by formulae (3.2) and (3.3).

Substituting expression (2.8) and (3.2) for the stress σ_{z2}, τ_{rz2} and the functions $p(\rho)$ and $q(\rho)$ into the equalities (4.1) and taking account of relation (2.9) and the fact that $C_2(\beta) \equiv D_2(\beta) \equiv 0$, we arrive at the SFE in $A_2(\beta)$ and $B_2(\beta)$

$$-k_{22}A_2 + n_2B_2 = \bar{p}(\beta), \quad m_2A_2 + s_2B_2 = \bar{q}(\beta) \tag{4.2}$$

The coefficients n_2, m_2 and s_2 are defined by formulae (2.10) for $i = 2$.

From system (4.2), we find expressions for the functions $A_2(\beta)$ and $B_2(\beta)$ and we substitute these into the general solution (2.8) when $i = 2$. As a result, we obtain the basic solution for the foundation layer. The representations of the axial and radial displacements $w_2(\rho, \zeta)$ and $u_2(\rho, \zeta)$ at the interface of the layer $\zeta = 0$, which are required for subsequent use, have the form

$$w_2(\rho, 0) = \int_0^\infty \Delta_{w2}(\beta)J_0(\rho\beta)d\beta, \quad u_2(\rho, 0) = \int_0^\infty \Delta_{u2}(\beta)J_1(\rho\beta)d\beta \tag{4.3}$$

$$\Delta_{\upsilon 2}(\beta) = E_2^{-1}(1+\nu_2)bD_{\upsilon 2}(\beta), \quad D_{\upsilon 2}(\beta) = \Delta_{\upsilon p2}(\beta)\bar{p}(\beta) + \Delta_{\upsilon q2}(\beta)\bar{q}(\beta)$$
$$\Delta_{\upsilon s2} = A_{\upsilon s2} = \frac{a_{\upsilon s2}(c)}{R_{l2}(c)}; \quad \upsilon = w, u; \quad s = p, q \tag{4.4}$$

The functions $a_{\upsilon s2}(c)$ and $R_{l2}(c)$ of the velocity c as a parameter are determined using formulae (3.12), taking account of the replacement of ν_1, k_{11} and k_{21} by ν_2, k_{12} and k_{22} respectively. All the properties of the functions $a_{\upsilon s1}$ and R_{l1}, which were mentioned at the end of Section 3 are extended to the functions $a_{\upsilon s2}$ ($\upsilon = w, u$; $s = p, q$) and R_{l2} in the interval $0 < c < c_{22}$. In particular, Eq. (3.14), when account is taken of the replacement of ν_1 and x_1 by ν_2 and $x_2 = c^2/c_{22}^2$, is the dispersion equation for the foundation layer (the half-space). The resonance velocity $c_s = c_{22}\sqrt{x_{2S}}$ of the natural non-dispersing Stoneley waves propagating along the interface of the layer and the half-space $\zeta = 0$ is determined in terms of the real root of this equation x_{2S}.

5. FORMULATION AND SOLUTION OF THE MIXED PROBLEM OF THE PROPAGATION OF A CRACK WITH A CAVITY AT THE TIP

The whole of the unbounded domain $L = (0 \le \rho < \infty)$ of the outer boundary plane $\zeta = 1$ is free from normal and shear stresses:

$$\sigma_{z1} = 0, \quad \tau_{rz1} = 0, \quad \rho \in L \tag{5.1}$$

The still unknown friction law $q_T(\rho)$ for the shear stresses, which are directed towards the motion (with a minus sign) and the conditions of continuity for the axial displacements

$$\tau_{rz1} = \tau_{rz2} = -q_T(\rho), \quad w_1 = w_2, \quad \rho \in L_1 \tag{5.2}$$

must be satisfied at the interface of the layers $\zeta = 0$ in the region of contact of the sides of the moving crack $L_1 = (0 \le \rho < \alpha^0)$.

In cavity domain $L_2 = (\alpha^0 < \rho < 1)$, the axial and shear stresses are specified

$$\sigma_{z1} = \sigma_{z2} = 0, \quad \tau_{rz1} = \tau_{rz2} = 0, \quad \rho \in L_2 \tag{5.3}$$

and, outside the crack in the domain $L_3 = (1 \le \rho < \infty)$, the conditions of continuity of the axial and radial displacements

$$w_1 = w_2, \quad u_1 = u_2, \quad \rho \in L_3 \tag{5.4}$$

and the normal and shear stresses

$$\sigma_{z1} = \sigma_{z2}, \quad \tau_{rz1} = \tau_{rz2}, \quad \rho \in L_3 \tag{5.5}$$

must be satisfied.

The basic solutions of Sections 3 and 4 automatically satisfy all the boundary conditions (5.1) and (5.5) in the case of arbitrary transformations $\bar{p}(\beta)$ and $\bar{q}(\beta)$ of the axial and shear stresses at the interface of the layers $\zeta = 0$, which are to be determined from the boundary conditions (5.2)–(5.4). The choice of the law of friction force $q_T(\rho)$ $(0 \le \rho < \alpha^0)$ must ensure the synthesis of all these conditions, by which we mean the reduction of the solution of the initial mixed problem (5.1)–(5.5) to a single uniquely solvable system of integral equations. The mathematical apparatus used here enables us to determine the function $q_T(\rho)$ uniquely.

It will next be established that it is the generalized Coulomb friction law

$$q_T(\rho) = \rho(\mu p_T(\rho) + Q), \quad \rho \in L_1 \tag{5.6}$$

which synthesizes the boundary conditions and, at the same time, satisfies the condition of axial symmetry. Here, $p_T(\rho)$ is the intensity of the pressure of the layers which are rubbing against one another in the region of contact of the sides of the moving crack, μ is the dimensionless constant coefficient of friction and Q is the tangential component of the constant bonding force of the layers. The friction law (5.6) will will required in the final stage of the derivation of the system of resolving integral equations but, for the present, we will begin the successive transformation of the boundary conditions (5.2)–(5.4) by assuming that the function $q_T(\rho)$ is arbitrary.

Substituting expressions (3.2), (3.8) and (4.3) for the integral representations of the stresses and the basic solutions for the displacements in the plane of the crack $\zeta = 0$ into the boundary conditions (5.2)–(5.4), we arrive at a system of three integral equations in the Hankel transforms $\bar{p}(\beta)$ and $\bar{q}(\beta)$

$$\int_0^\infty \beta\bar{q}(\beta)J_1(\rho\beta)d\beta = -q_T(\rho), \quad \int_0^\infty D_{w12}(\beta)J_0(\rho\beta)d\beta = 0, \quad \rho \in L_1 \tag{5.7}$$

$$\int_0^\infty \beta\bar{p}(\beta)J_0(\rho\beta)d\beta = 0, \quad \int_0^\infty \beta\bar{q}(\beta)J_1(\rho\beta)d\beta = 0, \quad \rho \in L_2 \tag{5.8}$$

$$\int_0^\infty D_{w12}(\beta)J_0(\rho\beta)d\beta = 0, \quad \int_0^\infty D_{u12}(\beta)J_1(\rho\beta)d\beta = 0, \quad \rho \in L_3 \tag{5.9}$$

where

$$D_{\upsilon 12}(\beta) = \Delta_{\upsilon p12}(\beta)\bar{p}(\beta) + \Delta_{\upsilon q12}(\beta)\bar{q}(\beta); \quad \upsilon = w, u \tag{5.10}$$

$$\Delta_{\upsilon s12}(\beta) = \Delta_{\upsilon s1}(\beta) - \chi\Delta_{\upsilon s2}(\beta); \quad \upsilon = w, u; \quad s = p, q \tag{5.11}$$

The functions $\Delta_{\upsilon s1}(\beta)$ and $\Delta_{\upsilon s2}(\beta)$ are defined in Sections 3 and 4 by formulae (3.9) and (4.4).

In the system of equations (5.7)–(5.9), we change from the initial transforms $\bar{p}(\beta)$ and $\bar{q}(\beta)$ to the new unknown transforms $\bar{f}(\beta)$ and $\bar{g}(\beta)$ using the formulae

$$\bar{f}(\beta) = D_{u12}(\beta), \quad \bar{g}(\beta) = D_{w12}(\beta) \tag{5.12}$$

corresponding to the local functions $f(\rho)$ and $g(\rho)$:

$$f(\rho) = \begin{cases} f_1(\rho), & \rho \in L_1 \\ f_2(\rho), & \rho \in L_2; \\ 0, & \rho \in L_3 \end{cases} \quad g(\rho) = \begin{cases} g_2(\rho), & \rho \in L_2 \\ 0, & \rho \in L_1 \cup L_3 \end{cases} \tag{5.13}$$

the transforms of which

$$\bar{f}(\beta) = \bar{f}_1(\beta) + \bar{f}_2(\beta), \quad \bar{g}(\beta) = \bar{g}_2(\beta) \tag{5.14}$$

are defined by the Hankel integrals

$$\bar{f}_k(\beta) = \int_{L_k} \rho f_k(\rho) J_0(\rho\beta) d\rho, \quad k = 1, 2; \quad \bar{g}_2(\beta) = \int_{L_2} \rho g_2(\rho) J_1(\rho\beta) d\rho \tag{5.15}$$

When account is taken of the constructive expressions for the functions $D_{\upsilon 12}(\beta)$ ($\upsilon = w, u$) (5.10), the equalities (5.12) are an SFE in the transforms $\bar{p}(\beta)$ and $\bar{q}(\beta)$. From this SFE, we find expressions for the initial transforms $\bar{p}(\beta)$ and $\bar{q}(\beta)$ in terms of the new transforms $\bar{f}(\beta)$ and $\bar{g}(\beta)$

$$\bar{p}(\beta) = N_u(\beta), \quad \bar{q}(\beta) = N_w(\beta) \tag{5.16}$$

$$N_\upsilon(\beta) = \Delta_{\upsilon f}(\beta)\bar{f}(\beta) + \Delta_{\upsilon g}(\beta)\bar{g}(\beta), \quad \upsilon = w, u \tag{5.17}$$

where

$$\Delta_{wf} = \frac{\Delta_{wp12}}{\Delta_{pq}}, \quad \Delta_{wg} = -\frac{\Delta_{up12}}{\Delta_{pq}}, \quad \Delta_{uf} = -\frac{\Delta_{wq12}}{\Delta_{pq}}, \quad \Delta_{ug} = \frac{\Delta_{uq12}}{\Delta_{pq}},$$
$$\Delta_{pq} = \Delta_{wp12}\Delta_{uq12} - \Delta_{wq12}\Delta_{up12} \tag{5.18}$$

Substituting expressions (5.12) and (5.16) into Eqs (5.7)–(5.9), we obtain the following system of three integral equations in the new transforms $\bar{f}(\beta)$ and $\bar{g}(\beta)$

$$\int_0^\infty \beta N_w(\beta) J_1(\rho\beta) d\beta = -q_T(\rho), \quad \int_0^\infty \bar{g}(\beta) J_0(\rho\beta) d\beta = 0, \quad \rho \in L_1 \tag{5.19}$$

$$\int_0^\infty \beta N_u(\beta) J_0(\rho\beta) d\beta = 0, \quad \int_0^\infty \beta N_w(\beta) J_1(\rho\beta) d\beta = 0, \quad \rho \in L_2 \tag{5.20}$$

$$\int_0^\infty \bar{g}(\beta) J_0(\rho\beta) d\beta = 0, \quad \int_0^\infty \bar{f}(\beta) J_1(\rho\beta) d\beta = 0, \quad \rho \in L_3 \tag{5.21}$$

Next, we carry out the following transformations of Eqs (5.19)–(5.21). We integrate the first equation with respect to ρ in the limits from 0 to ρ and differentiate the second equation with respect to ρ. We multiply the first equation of (5.20) by ρ and integrate with respect to ρ within the limits from α^0 to ρ and then divide by ρ. The second equation is integrated with respect to ρ within the limits from α^0 to ρ, the first equation of (5.21) is differentiated with respect to ρ and the second equation is multiplied by ρ, differentiated with respect to ρ and then divided by ρ. As a result, the transformed equations acquire the form

$$\int_0^\infty N_w(\beta) J_0(\rho\beta) d\beta = F(\rho) + C_1, \quad \int_0^\infty \beta \bar{g}(\beta) J_1(\rho\beta) d\beta = 0, \quad \rho \in L_1 \tag{5.22}$$

$$\int_0^\infty N_u(\beta) J_1(\rho\beta) d\beta = \frac{D_2}{\rho}, \quad \int_0^\infty N_w(\beta) J_0(\rho\beta) d\beta = C_2, \quad \rho \in L_2 \tag{5.23}$$

$$\int_0^\infty \beta \bar{g}(\beta) J_1(\rho\beta) d\beta = 0, \quad \int_0^\infty \beta \bar{f}(\beta) J_0(\rho\beta) d\beta = 0, \quad \rho \in L_3 \tag{5.24}$$

where

$$F(\rho) = \int_0^\rho q_T(x) dx \tag{5.25}$$

and C_1, C_2 and D_2 are arbitrary constants which are determined below. According to the theorem for the inversion of Hankel transforms $\bar{f}(\beta)$ and $\bar{g}(\beta)$ (5.14), (5.15), the second equation of (5.22) and both equations of (5.24) turn into identities, and the latter are therefore excluded from further consideration. We emphasize that future references to formulae (5.22) refer only to the first equation, while the identity has played its role and is no longer required.

When account is taken of formulae (5.14), the remaining equations (5.22) and (5.23) in the bounded mixed contours $L_1 = (0 \le \rho < \alpha^2)$, $L_2 = (\alpha^0 < \rho < 1)$ form a closed system for determining the unknown transforms $\bar{f}_1, \bar{f}_2$ and $\bar{g}_2$ (5.15). For the further transformation of this system it is necessary to separate out the leading terms of the functions (5.18) at infinity when $\beta \to \infty$

$$\Delta_{\upsilon r}(\beta) = A_{\upsilon r} + \Delta^*_{\upsilon r}(\beta); \quad \upsilon = w, u; \quad r = f, g \tag{5.26}$$

where

$$A_{wf} = \frac{A_{wp12}}{A_{pq}}, \quad A_{wg} = -\frac{A_{up12}}{A_{pq}}; \quad A_{uf} = -\frac{A_{wq12}}{A_{pq}}, \quad A_{ug} = \frac{A_{uq12}}{A_{pq}}$$

$$A_{pq} = A_{wp12}A_{uq12} - A_{wq12}A_{up12} \tag{5.27}$$

$$A_{wp12} = -(A_{wp1} + \chi A_{wp2}), \quad A_{wq12} = A_{wq1} - \chi A_{wq2}$$

$$A_{up12} = A_{up1} - \chi A_{up2}, \quad A_{uq12} = -(A_{uq1} + \chi A_{uq2})$$

The quantities $A_{\upsilon s1}$ and $A_{\upsilon s2}$ ($\upsilon = w, u$; $s = p, q$) are irrational algebraic functions of the velocity c, which are defined by formulae (3.12) and (4.4).

When $\beta \to \infty$, the functions $\Delta^*_{\upsilon r}(\beta) = \Delta_{\upsilon r}(\beta) - A_{\upsilon r}$ which are determined using formulae (5.26), are of the order of infinitesimal functions:

$$\Delta^*_{\upsilon r}(\beta) = O(\exp(-2\lambda k_2 \beta)); \quad \upsilon = w, u; \quad r = f, g; \quad k_2 = \min(k_{21}, k_{22}) \tag{5.28}$$

The special mathematical apparatus for investigating an analogous system of integral equations in the fundamental mixed problem of the theory of elasticity can be completely extended to the system of integral equations (5.22), (5.23) in expanded form, taking account of the representations of the functions $\bar{f}(\beta)$, $\bar{g}(\beta)$, $N_\upsilon(\beta)$ and $\Delta_{\upsilon r}(\beta)$ and the asymptotic form $\Delta^*_{\upsilon r}(\beta)$ using formulae (5.14), (5.17), (5.26) and (5.28).† On applying it and, at the same time, taking account of the synthesizing friction law $q_T(\rho)$ (5.6) in formula $F(\rho)$ (5.25), we reduce the expanded system of integral equations (5.22), (5.23) for the transforms $\bar{f}_1(\beta), \bar{f}_2(\beta)$ and $\bar{g}_2(\beta)$ (5.15) to the following system of three singular integral equations (SIE) with Cauchy kernels for functions of the real variable $\varphi_j(x)$ ($j = 1, 2, 3$) for the initial mixed problem of a moving crack

† NIKISHIN, V. S., The correct formulation and numerical solution of fundamental and mixed problems in the theory of elasticity for multilayer and continuously inhomogeneous media. Doctorate Dissertation, 01.01.07, Vychisl. Tsentr, Akad Nauk SSSR, Moscow, 1982.

$$A_{wp12}\varphi_1(x) - \frac{A_{wq12}\mu x}{\pi}\int_{-\alpha^0}^{\alpha^0}\frac{\varphi_1(t)}{t-x}dt + \frac{A_{pq}}{\pi}\int_{-\alpha^0}^{\alpha^0}K_{11}(x,t)\varphi_1(t)dt +$$
$$+\frac{2A_{pq}}{\pi}\int_{\alpha^0}^{1}\frac{K_{12}(x,t)\varphi_2(t)+K_{13}(x,t)\varphi_3(t)}{\sqrt{t-\alpha^0}}dt = \frac{2A_{pq}}{\pi}(Qx^2+C_1), \quad -\alpha^0 \le x \le \alpha^0 \tag{5.29}$$

$$A_{wp12}\varphi_2(x) - \frac{A_{up12}}{\pi}\int_{\alpha^0}^{1}\frac{\varphi_3(t)}{t-x}dt + \chi_2(\varphi_1,\varphi_2,\varphi_3;x) = \frac{2A_{pq}}{\pi}\frac{C_2 x}{\sqrt{x^2-\alpha^{02}}}, \quad \alpha^0 < x \le 1 \tag{5.30}$$

$$A_{uq12}\varphi_3(x) + \frac{A_{wq12}}{\pi}\int_{\alpha^0}^{1}\frac{\varphi_2(t)}{t-x}dt + \chi_3(\varphi_1,\varphi_2,\varphi_3;x) = \frac{2A_{pq}}{\pi}\frac{D_2}{\sqrt{x^2-\alpha^{02}}}, \quad \alpha^0 < x \le 1 \tag{5.31}$$

Here

$$\chi_j(\varphi_1,\varphi_2,\varphi_3;x) = \frac{2A_{pq}}{\pi}\left(\int_0^{\alpha^0}\frac{K_{j1}(x,t)\varphi_1(t)}{\sqrt{x-\alpha^0}}dt + \right.$$
$$\left.+\int_{\alpha^0}^{1}\frac{K_{j2}(x,t)\varphi_2(t)+K_{j3}(x,t)\varphi_3(t)}{\sqrt{(x-\alpha^0)(t-\alpha^0)}}dt\right), \quad j = 2,3$$

$$K_{11} = G_{0011} - \mu x G_{1011}, \quad K_{12} = A_{wf}t\eta_1(x,t) + G_{0112} - \mu x t G_{1012}$$
$$K_{13} = A_{wg}/\sqrt{\alpha^0+t} + G_{0113} - \mu x(A_{ug}x\eta_1(x,t) + G_{1113}) \tag{5.32}$$
$$K_{21} = A_{wf}x\eta_1(x,t) + xG_{0021}, \quad K_{22} = xtG_{0022} + A_{wf}M_{22}, \quad K_{23} = xG_{0123} + A_{wg}M_{23}$$
$$K_{31} = A_{uf}/\sqrt{\alpha^0+x} + G_{1031}, \quad K_{32} = tG_{1032} + A_{ug}M_{32}, \quad K_{33} = G_{1133} + A_{ug}M_{33}$$

$$G_{kmjn} = \int_0^\infty b_{kmjn}S_{kj}(x,\beta)S_{mn}(t,\beta)d\beta; \quad k,m = 0,1; \quad j,n = 1,2,3$$

$$b_{0011} = b_{0021} = b_{0022} = b_{0112} = \Delta^*_{wf}(\beta), \quad b_{0113} = b_{0123} = \Delta^*_{wg}(\beta)$$

$$b_{1011} = b_{1012} = b_{1031} = b_{1032} = \Delta^*_{uf}(\beta), \quad b_{1113} = b_{1133} = \Delta^*_{ug}(\beta)$$

$$S_{01}(x,\beta) = \cos(x\beta), \quad S_{11}(x,\beta) = \sin(x\beta)$$

$$S_{0j}(x,\beta) = \frac{J_0(\alpha^0\beta)}{\sqrt{x+\alpha^0}} - \beta\sqrt{x-\alpha^0}\int_{\alpha^0}^{x}\frac{J_1(\rho\beta)}{\sqrt{x^2-\rho^2}}d\rho, \quad j = 2,3$$
$$S_{1j}(x,\beta) = \frac{\alpha^0 J_1(\alpha^0\beta)}{\sqrt{x+\alpha^0}} + \beta\sqrt{x-\alpha^0}\int_{\alpha^0}^{x}\frac{\rho J_0(\rho\beta)}{\sqrt{x^2-\rho^2}}d\rho, \quad j = 2,3 \tag{5.33}$$
$$M_{jj} = \frac{2m_{jj}(x,t)}{\pi\eta_2^+(x,t)}, \quad j = 2,3; \quad M_{23} = \frac{\eta_2^-(x,t)}{t-x}\eta_3(x,t), \quad M_{32} = -\frac{\eta_2^-(x,t)}{t-x}\eta_3(t,x)$$

$$m_{22} = \frac{xt}{2(x^2-t^2)}\left(\frac{1}{x}\eta_4(x) - \frac{1}{t}\eta_4(t)\right), \quad m_{33} = -\alpha^0 + \frac{1}{2(x^2-t^2)}(x\eta_4(x) - t\eta_4(t))$$

$$\eta_1(x,t) = \frac{1}{t^2-x^2}\sqrt{\frac{\alpha^{02}-x^2}{\alpha^0+t}}, \quad \eta_2^{\pm}(x,t) = \sqrt{(x\pm\alpha^0)(t\pm\alpha^0)}$$

$$\eta_3(x,t) = \frac{x}{t+x}\sqrt{\frac{t^2-\alpha^{02}}{x^2-\alpha^{02}}} - \frac{1}{2}, \quad \eta_4(x,t) = (x^2-\alpha^{02})\ln\frac{x+\alpha^0}{x-\alpha^0}$$

Note that the functions $M_{ij}(x, t)$ (5.33), which are defined in the square $\alpha^0 \le xt \le 1$, have removable mobile singularities on the diagonal $t = x$. The values of these functions when $t = x$ are assumed to be equal to their limit values when $t \to x$, which are determined by l'Hôpital's rule.

The unknown transforms $\bar{f}_1(\beta), \bar{f}_2(\beta)$ and $\bar{g}_2(\beta)$ (5.15) are expressed in terms of the functions $\varphi_j(x)$ ($j = 1, 2, 3$), which satisfy the system of SIE (5.29)–(5.31), using the formulae

$$\bar{f}_1(\beta) = \int_0^{\alpha^0}\varphi_1(x)\cos(x\beta)dx, \quad \bar{f}_2(\beta) = \int_{\alpha^0}^{1}\frac{x\varphi_2(x)S_{02}(x,\beta)}{\sqrt{x-\alpha^0}}dx$$
$$\bar{g}_2(\beta) = \int_{\alpha^0}^{1}\frac{\varphi_3(x)S_{12}(x,\beta)}{\sqrt{x-\alpha^0}}dx \tag{5.34}$$

It follows automatically from the theory of an identity transformation of the system of equations (5.22), (5.23) for the transforms $\bar{f}(\beta) = \bar{f}_1(\beta) + \bar{f}_2(\beta)$ and $\bar{g}(\beta) = \bar{g}_2(\beta)$ (5.15) into the system of SIE (5.29)–(5.31) that the transforms $\bar{f}(\beta)$ and $\bar{g}(\beta)$, when account is taken of formulae (5.34), turns all Eqs (5.22) and (5.23) and the initial system (5.19)–(5.21) into identities, with the possible exception of Eqs (5.21). During the transformation process, these last equations are differentiated with respect to ρ and are therefore subject to verification by substituting the transforms $\bar{f}_1(\beta), \bar{f}_2(\beta)$ and $\bar{g}_2(\beta)$ (5.34) into them. Verification showed that Eqs (5.21) are only satisfied subject to the additional conditions

$$\int_0^{\alpha^0}\varphi_1(x)dx = 0, \quad \int_{\alpha^0}^{1}\frac{x\varphi_2(x)}{\sqrt{x^2-\alpha^{02}}}dx = 0, \quad \int_{\alpha^0}^{1}\frac{\varphi_3(x)}{\sqrt{x^2-\alpha^{02}}}dx = 0 \tag{5.35}$$

from which the arbitrary constants C_1, C_2 and D_2 on the right-hand side of the system if SIE (5.29)–(5.31) are determined.

In order to satisfy conditions (5.35), we will seek a solution of the system of SIE (5.29)–(5.31) in the form

$$\varphi_j(x) = \varphi_{j1}(x) + C_1\varphi_{j2}(x) + C_2\varphi_{j3}(x) + D_2\varphi_{j4}(x), \quad j = 1, 2, 3 \tag{5.36}$$

where $\varphi_{jn}(x)$ ($n = 1, 2, 3, 4$) are particular solutions of the system when account is taken of the following equalities on its right-hand side respectively: (1) $C_1 = C_2 = D_2 = 0$, (2) $Q = 0, C_1 = 1, C_2 = D_2 = 0$, (3) $Q = 0, C_2 = 1, C_1 = D_2 = 0$, (4) $Q = 0, D_2 = 1, C_1 = C_2 = 0$.

Apart from the constants C_1, C_2 and D_2 which have been determined above, there is also a theoretically indeterminate constant $Q > 0$ on the right-hand side of the system of SIE (5.29)–(5.31), which is the magnitude of the tangential force directed towards the motion of the crack, which depends on the strength of the bonding (adhesion) of the surfaces of the different layers. However, the structure of the system of SIE (5.29)–(5.31) enables us to construct a solution in the case of an arbitrary constant $Q > 0$ in the form

$$\varphi_j(x) = Q\varphi_j^0(x), \quad j = 1, 2, 3 \tag{5.37}$$

where $\varphi_j^0(x)$ ($j = 1, 2, 3$) is the particular solution when $Q = 1$. In passing, the hypothetical case when $Q = 0$ should be noted. In this case, the system of SIE (5.29)–(5.31) has only a trivial solution

$\varphi_j^0(x) \equiv 0$ ($j = 1, 2, 3$) which is evidenced of the degeneration of the initial problem and thereby underlines the special role of the force Q in the synthesizing friction law (5.6).

Next, we shall briefly describe a method for constructing a solution of the SIE (5.29)–(5.31) and investigate its singularities. The process begins with the construction and investigation of the singularities of the solution of the characteristic system of SIE (5.29)–(5.31) in the complex form

$$A_{wp12}\varphi_1(x) - i\frac{A_{wq12}\mu x}{\pi i}\int_{-\alpha^0}^{\alpha^0}\frac{\varphi_1(t)}{t-x}dt = \frac{2A_{pq}}{\pi}(Qx^2 + C_1), \quad -\alpha^0 \le x \le \alpha^0 \tag{5.38}$$

$$A\varphi(x) + \frac{B}{\pi i}\int_{\alpha^0}^{1}\frac{\varphi(t)}{t-x}dt = A_{pq}f(x), \quad \alpha^0 \le x \le 1 \tag{5.39}$$

Here

$$A = \left\| \begin{matrix} A_{wp12} & 0 \\ 0 & A_{uq12} \end{matrix} \right\|, \quad B = \left\| \begin{matrix} 0 & -iA_{up12} \\ iA_{wq12} & 0 \end{matrix} \right\|, \quad \varphi(x) = \left\| \begin{matrix} \varphi_2(x) \\ \varphi_3(x) \end{matrix} \right\|$$
$$f(x) = \left\| \begin{matrix} f_2(x) \\ f_2(x) \end{matrix} \right\|, \quad f_j(x) = \frac{f_j^0(x)}{\pi\sqrt{x-\alpha^0}}, \quad j = 2, 3; \quad f_2^0(x) = \frac{C_2 x}{\sqrt{x+\alpha^0}}, \quad f_3^0(x) = \frac{D_2}{\sqrt{x+\alpha^0}} \tag{5.40}$$

Equations (5.38) and (5.39) are mutually independent, possess zero indices and have unique closed analytical solutions, which are constructed using well-known theory [3, 4]. At the same time, Eq. (5.39), (5.40) in matrix form reduces to a Riemann–Hilbert problem. During the process of constructing the closed analytical solutions of the characteristic SIE (5.38) and (5.39), (5.40), their singularities at the ends of the intervals of integration $x = \alpha^0$ and $x = 1$ manifest themselves and are separated out in explicit form

$$\begin{aligned} &\varphi_1^+(x) \sim c^+\ln(\alpha^0 - x), \quad \varphi_1^-(x) \sim c^-(\alpha^0 - x)^{-\theta} \quad \text{when} \quad x \to \alpha^0 - 0 \\ &\varphi_j^+(x) \sim c\ln(1-x), \quad j = 2, 3 \quad \text{when} \quad x \to 1 - 0 \end{aligned} \tag{5.41}$$

The superscripts on the functions $\varphi_1^\pm(x)$ correspond to the signs of the ratio $\xi = A_{wq12}/A_{wp12} = \pm|\xi|$ over the range of variation of the mechanical characteristics of the problem, the constant θ is defined by formula (7.7) in Appendix 1 and the actual values of the constants $c^\pm$ and c are no longer required.

The closed analytical solution of the characteristic SIE (5.38), (5.3) is used to regularize the overall system of SIE (5.29)–(5.31) using the Carleman–Vekua method. Here, the regular kernels and the free terms at the ends $x = \alpha^0$, 1 preserve the root singularities and acquire new singularities (5.41) which, after they have been separated out, are removed together with root singularities by means of identity transformations. As a result of regularization, the system of SIE (5.29)–(5.31) reduces to a uniquely solvable system of three regular Fredholm integral equations of the second and third kind with continuous kernels in the functions $\Phi_j(x)$ ($j = 1, 2, 3$) which are connected with the initial functions $\varphi_j(x)$ ($j = 1, 2, 3$) if the initial system of SIE by the relations

$$\varphi_1^+ = \Phi_1^+\ln(\alpha^0 - x), \quad \varphi_1^- = \Phi_1^-(\alpha^0 - x)^{-\theta}, \quad \varphi_j^\pm = \Phi_j^\pm\ln(1-x), \quad j = 2, 3 \tag{5.42}$$

Substituting the transform $\bar{p}(\beta)$, expressed using formulae (5.16), (5.17) and (5.14) in terms of the transforms $\bar{f}_1(\beta)$, $\bar{f}_2(\beta)$ and $\bar{g}_2(\beta)$ (5.34), into the Hankel integral for $\sigma_z(\rho) = p(\rho)$ (3.2), we carry out a series of successive transformations of the double integrals and, as a result, when account is taken of relations (5.42), we determine the leading terms of the normal stresses $\sigma_z(\rho)$ on the contours of the cavity $\alpha^0 < \rho < 1$

$$\sigma_z(\rho) = \frac{A_{uf}\Phi_1^\pm(\alpha^0)\gamma_1^\pm(\rho, \alpha^0) + A_{ug}\Phi_3^\pm(\alpha^0)\gamma_3(\alpha^0)}{\sqrt{\alpha^{02} - \rho^2}} + O(1) \quad \text{when} \quad \rho \to \alpha^0 - 0 \tag{5.43}$$

$$\sigma_z(\rho) = -\frac{A_{ug}\Phi_3^{\pm}(1)\ln|1-\rho|}{\sqrt{\rho^2-1}} + O(1) \quad \text{when} \quad \rho \to 1+0 \tag{5.44}$$

where

$$\gamma_1^+ = \ln(\alpha^{02}-\rho^2), \quad \gamma_1^- = \frac{C^-(1-2\theta)}{2^\theta(\alpha^{02}-\rho^2)^\theta}, \quad \gamma_3 = \frac{1}{2}\alpha^0\ln(1-\alpha^0)$$
$$0 < C^- = \int_0^{\alpha^0}\frac{dt}{(\alpha^{02}-t^2)^\theta} < \infty \tag{5.45}$$

Taking account of the inequality $\ln|1-\rho| < 0$, we conclude from the limiting formulae (5.43)–(5.45) that the criteria for the existence of a cavity $\alpha^0 < \rho < 1$ for the two forms of the solution reduce to the following conditions (see Section 1)

$$\Phi_1^2(\alpha^0) + \Phi_3^2(\alpha^0) = 0, \quad A_{ug}\Phi_3(1) > 0 \tag{5.46}$$

under which the normal stresses $\sigma_z(\rho)$ on the internal contour $\rho = \alpha^0$ are bounded and the stresses on the contour of the tip of the crack $\rho = 1$ are tensile stresses (of positive sign) and they undergo an infinite discontinuity $\sigma_z(1) = +\infty$. By analysing formulae (5.27), (3.12), (3.13) and (4.4), it can be shown that the quantity A_{ug}, as a function of the velocity c, changes sign at the point of resonance $c = c_R$ of the Rayleigh waves in the case when $c_{21} < c_{22}$ or at the point $c = c_S$ of the Stoneley waves in the case when $c_{22} < c_{21}$. In both cases, we have $A_{ug} < 0$ in the preresonance velocity interval $0 < c < \min(c_R, c_S)$ and $A_{ug} > 0$ in the post-resonance velocity interval $\min(c_R, c_S) < c < \min(c_{21}, c_{22})$. In the case of the resonance of Rayleigh or Stoneley waves when $c \to \min(c_R \mp 0, c_s \mp 0)$, we have $A_{ug} \to \mp\infty$.

It should be explained that the transition to resonance of Rayleigh or Stoneley waves when $c \to c_\upsilon$ ($\upsilon = r, s$) is accompanied by a secular increase in the amplitude A of the oscillation of all of the stresses and displacements according to the hyperbolic law $A = K/(c - c_\upsilon)$ (K = const) and, when $c = c_\upsilon$, it is concluded with a catastrophic global rupture during which all of the stresses and displacements become infinite with an instantaneous change of sign.

The magnitude of the internal radius of the cavity α^0 is determined numerically on the basis of the criterion (5.46) and the parameters of problem (1.1), (1.2), corresponding to this case, are chosen.

In order to substantiate the correctness of the solution constructed above, apart from satisfying the criterion (5.46), it is required that the continuous axial stress $\sigma_z(\rho)$ on the contacting sides of the crack should be compressive stresses (of negative sign) $\sigma_z(\rho) < 0$ $(0 \le \rho \le \alpha^0)$. The latter condition, which must indicate that there are no intermediate cavities (Section 1), can only be verified numerically.

6. THE PROBLEM OF THE MOTION OF A TRANSVERSE SHEAR CRACK WITHOUT A CAVITY

The problem of the motion of a transverse shear crack without a cavity is considered as a special case of the problem from Section 5 in which

$$\alpha^0 \equiv 1, \quad \bar{f}_2(\beta) \equiv \bar{g}_2(\beta) \equiv 0, \quad \varphi_2(x) \equiv \varphi_3(x) \equiv 0 \tag{6.1}$$

Below, in this section when reference is made to the formulae in Section 5, the special case of these formulae which corresponds to conditions (6.1) being satisfied should always be kept in mind.

The problem being considered is reduced to the SIE (5.29) in the function $\varphi_1(x)$ in the interval [0, 1]. The transform $\bar{f}_1(\beta)$ is expressed in terms of $\varphi_1(x)$ using formula (5.34). The constant C_1 is determined from the additional condition (5.35) for $\varphi_1(x)$. In order to satisfy this condition, we will seek a solution of the SIE (5.29) in the form

$$\varphi_1(x) = \varphi_{11}(x) + C_1\varphi_{12}(x) \tag{6.2}$$

where $\varphi_{1j}(x)$ (j = 1, 2) are the particular solutions of the SIE (5.29) when (1) $C_1 = 0$, and (2) $Q = 0$, $C_1 = 1$ respectively. On substituting the function $\varphi_1(x)$ into equality (5.5), we obtain an equation in C_1 from which we find

$$C_1 = \int_0^1 \varphi_{11}(x)dx\left(\int_0^1 \varphi_{12}(x)dx\right)^{-1} \quad (6.3)$$

The method of constructing the solution of the SIE (5.29) and investigating its singularities has been described in detail in Section 5. The closed solution of the characteristic SIE (5.38) is constructed in two forms $\varphi_1^{\pm}(x)$, which have the singularities (5.41) when $x \to 1$. The complete SIE (5.29), as a result of regularization, reduces to a Fredholm integral equation of the second kind in two forms in the continuous functions $\Phi_1^{\pm}(x)$, which are related to the solutions $\varphi_1^{\pm}(x)$ of the initial SIE (5.29) using formulae (5.42).

The leading terms of the stresses $\sigma_z(\rho)$ on the contour of the crack tip $\rho = 1$ for the two forms of the solution are determined using formula (5.44)

$$\sigma_z(\rho) = \frac{A_{uf}\Phi_1^{\pm}(1)\gamma_1^{\pm}(\rho, 1)}{\sqrt{1-\rho^2}} + O(1) \text{ when } \rho \to 1-0 \quad (6.4)$$

The functions $\gamma_1^{\pm}(\rho, 1)$ are determined using formula (5.45).

Taking the equality $\gamma_1^{\dagger} = \ln(1-\rho^2) < 0$ into account, we conclude from the limiting formula (6.4) that the necessary criterion of the correctness of the solution which has been constructed reduces to the conditions

$$A_{uf}\Phi_1^{+}(1) > 0, \quad A_{uf}\Phi_1^{-}(1) < 0 \quad (6.5)$$

$$\sigma_z(\rho) < 0, \quad 0 \le \rho < 1 \quad (6.6)$$

The parameters of problem (1.1), (1.2) are selected from conditions (6.5), and condition (6.6) must numerically confirm that the stresses $\sigma_z(\rho)$ on the sides of the crack $0 \le \rho < 1$ are compressive stresses (Section 1).

As a function of c, the sign of $A_{uf}(c)$, unlike the sign of $A_{ug}(c)$ in Section 5, depends very much on the magnitude of the parameter χ (1.1) compared with the magnitude of the ratio $A_{12}(c^0) = A_{wq1}(c^0)/A_{wq2}(c^0)$, where c^0 is an arbitrarily selected preresonance velocity $0 < c^0 < \min(c_R, c_S)$. In the case when $c_{21} < c_{22}$: (1) when $\chi > A_{12}(c_0)$ $(0 < c^0 < c_R)$, we have $A_{uf}(c) > 0$ $(0 < c < c^0)$, $A_{uf}(c) < 0$ $(c^0 < c < c_R)$ and (2) when $\chi < A_{12}(c^0)$, we have $A_{uf}(c) < 0$ $(0 < c < c_R)$. In the case when $c_{22} < c_{21}$: (1) when $\chi > A_{12}(c^0)$ $(0 < c^0 < c_S)$, we have $A_{uf}(c) > 0$ $(0 < c < c_S)$ and (2) when $\chi < A_{12}(c^0)$, we have $A_{uf}(c) < 0$ $(0 < c < c^0)$, $A_{uf}(c) > 0$ $(c^0 < c < c_S)$. In the case of resonance $A_{uf}(c) \to \mp\infty$ when $c \to c_R \mp 0$ and $A_{uf}(c) = \pm\infty$ when $c \to c_S \mp 0$.

The resonance of Rayleigh and Stoneley waves in this solution has exactly the same form as in the solution of the problem in Section 5.

7. APPENDIX 1. THE CLOSED SOLUTION OF THE CHARACTERISTIC SYSTEM OF SIE (5.38)–(5.40)

We will now construct the solution of Eq. (5.38). Initially, in order to simplify the calculations, we will change from Eq. (5.38) to the equivalent equation

$$a(x)\varphi_1(x) + \frac{b(x)}{\pi i}\int_{-\alpha^0}^{\alpha^0} \frac{\varphi_1(t)}{t-x}dt = \tilde{f}_1(x), \quad -\alpha^0 \le x \le \alpha^0 \quad (7.1)$$

where

$$a(x) = \frac{A_{wp12}}{\Delta(x)}, \quad b(x) = -\frac{iA_{wq12}\mu x}{\Delta(x)}, \quad \tilde{f}_1(x) = \frac{2A_{pq}}{\pi\Delta(x)}(Qx^2 + c_1)$$
$$\Delta^2(x) = A_{wp12}^2 + (A_{wq12}\mu x)^2, \quad a^2(x) - b^2(x) = 1 \quad (7.2)$$

We will seek the solution of SIE (7.1) in the following form [3]

$$\varphi_{1h}(x) = a(x)\tilde{f}_1(x) - \frac{b(x)Z(x)}{\pi i}\int_{-\alpha^0}^{\alpha^0}\frac{\tilde{f}_1(t)dt}{Z(t)(t-x)} + b(x)Z(x)P_{\kappa-1}(x) \tag{7.3}$$

The solution of the characteristic SIE (7.1) is denote by $\varphi_{1h}(x)$, and $Z(x)$ is the canonical function of this equation

$$Z(x) = (x+\alpha^0)^{\lambda_1}(x-\alpha^0)^{\lambda_2}\exp(\Gamma(x))$$
$$\Gamma(x) = \frac{1}{2\pi i}\int_{-\alpha^0}^{\alpha^0}\frac{\ln G(t)}{t-x}dt, \quad G(x) = \frac{a(x)-b(x)}{a(x)+b(x)} \tag{7.4}$$

λ_1 and λ_2 are integers, which satisfy the conditions

$$-1 \le \alpha_k + \lambda_k < 1, \quad k = 1, 2$$
$$\alpha_1 = \mathrm{Re}\left(-\frac{\ln G(-\alpha^0)}{2\pi i}\right), \quad \alpha_2 = \mathrm{Re}\left(\frac{\ln G(\alpha^0)}{2\pi i}\right) \tag{7.5}$$

$\kappa = -(\lambda_1 + \lambda_2)$ is the index of SIE (7.1). $P_{\kappa-1}$ is a polynomial of degree $\kappa - 1$ with arbitrary coefficients and, when $\kappa \le 0$, it is necessary to put $P_{\kappa-1}(x) \equiv 0$.

Using formula (7.4) and taking account of relation (7.2), we obtain

$$G(x) = \frac{1+i\xi\mu x}{1-i\xi\mu x}, \quad \xi = \frac{A_{wq12}}{A_{wp12}}, \quad |G(x)| = 1, \quad \ln G(x) = 2i\,\mathrm{arctg}(\xi\mu x) \tag{7.6}$$

By analysing the quantities $A_{wq12} = A_{wq1} - \chi A_{wq2}$ and $A_{wp12} = -(A_{wp1} + \chi A_{wp2})$ as functions of $c < \min(c_{12}, c_{22})$, it can be established that their ratio ξ changes sign over the range of variation of the elastic and velocity characteristics of the two-layer half-space. Consequently, it is necessary to seek two various of the solution of Eq. (7.1): when $\xi > 0$ and when $\xi < 0$. For both versions, the solutions $\varphi_{1h}^{\pm}(x)$ (7.3), which are given the superscripts $\pm$ respectively, we find using formulae (7.4)–(7.6) that

$$\Gamma(x) = \pm\tilde{\Gamma}(x), \quad \tilde{\Gamma}(x) = \frac{1}{\pi}\int_{-\alpha^0}^{\alpha^0}\mathrm{arctg}(|\xi|\mu t)\frac{dt}{t-x}$$
$$\alpha_1 = \alpha_2 = \pm\theta, \quad \theta = \frac{1}{\pi}\mathrm{arctg}(|\xi|\mu\alpha^0), \quad 0 < \theta < \frac{1}{2} \tag{7.7}$$
$$\lambda_1 = \lambda_2 = 0, \quad \kappa = -(\lambda_1 + \lambda_2) = 0$$

In both cases when $\xi > 0$ and $\xi < 0$, the characteristic SIE (7.1) has zero index and its closed versions of the solution $\varphi_{1h}^{\pm}(x)$ are determined using formula (7.3), taking account of the relations and the identity $P_{\kappa-1}(x) \equiv 0$. We have

$$\varphi_{1h}^{\pm}(x) = \frac{A_{wp12}}{\Delta(x)}\tilde{f}_1(x) + \frac{A_{wq12}\mu x\exp(\pm\tilde{\Gamma}(x))}{\pi\Delta(x)}\int_{-\alpha^0}^{\alpha^0}\frac{\tilde{f}_1(t)\exp(\mp\tilde{\Gamma}(t))}{t-x}dt \tag{7.8}$$

We next investigate and separate out in an explicit form the singularities of the solutions $\varphi_{1h}^{\pm}(x)$ (7.8) at the ends $\pm\alpha^0$ of the interval of integration. For this purpose, we introduce the continuous and bounded functions in the interval $-\alpha^0 \le x \le \alpha^0$

$$\alpha(x) = \frac{1}{\pi}\mathrm{arctg}(|\xi|\mu x), \quad \gamma(x) = \int_{-\alpha^0}^{\alpha^0}\frac{\alpha(t)-\alpha(x)}{t-x}dt \tag{7.9}$$

into the treatment and, taking into account the functions

$$\pm\tilde{\Gamma}(x) = \pm\gamma(x) \pm \alpha(x)\ln\frac{\alpha^0 - x}{\alpha^0 + x}$$

which are expressed in terms of them, we obtain the representation of the canonical functions

$$Z^{\pm}(x) = \exp(\pm\tilde{\Gamma}(x)) = \exp(\pm\gamma(x))\left(\frac{\alpha^0 - x}{\alpha^0 + x}\right)^{\pm\alpha(x)} \tag{7.10}$$

Taking into account the continuity of the functions $\alpha(x)$ and $\gamma(x)$ (7.9), we establish that the function $Z^+(x)$ (7.10) is continuous in the interval $-\alpha^0 \leq x \leq \alpha^0$ and vanishes at its ends:

$$Z^+(\pm\alpha^0) = 0 \tag{7.11}$$

The function $Z^-(x)$ is continuous in the interval $-\alpha^0 < x < \alpha^0$, and, at its ends when $x \to \pm\alpha^0$, it undergoes infinite discontinuities according to the law

$$Z^-(x) = \exp(-\tilde{\Gamma}(x)) \sim \tilde{c}(\alpha^0 \mp x)^{-\theta}, \quad \tilde{c} = \exp(-\gamma(\alpha^0)) \tag{7.12}$$

We now separate out the singularities of the closed solutions $\varphi^{\pm}_{1h}(x)$ (7.8) of the characteristic SIE (7.1). In the case of $\varphi^{+}_{1h}(x)$, we represent the integral in (7.8) in the form

$$\int_{-\alpha^0}^{\alpha^0} \frac{\tilde{f}_1(t)\exp(-\tilde{\Gamma}(t))}{t-x}dt = \int_{-\alpha^0}^{\alpha^0} \frac{\tilde{f}_1(t) - \tilde{f}_1(x)}{t-x}\exp(-\tilde{\Gamma}(t))dt +$$

$$+ \tilde{f}_1(x)\exp(-\tilde{\Gamma}(x))\left(\ln\frac{\alpha^0 - x}{\alpha^0 + x} - \int_{-\alpha^0}^{\alpha^0} \frac{\exp(\tilde{\Gamma}(t)) - \exp(\tilde{\Gamma}(x))}{t-x}\exp(-\tilde{\Gamma}(t))dt\right) \tag{7.13}$$

and, in the case of $\varphi^{-}_{1h}(x)$, in the form

$$\int_{-\alpha^0}^{\alpha^0} \frac{\tilde{f}_1(t)\exp(\tilde{\Gamma}(t))}{t-x}dt = \int_{-\alpha^0}^{\alpha^0} \frac{\tilde{f}_1(t)\exp(\tilde{\Gamma}(t)) - \tilde{f}_1(x)\exp(\tilde{\Gamma}(x))}{t-x}dt +$$

$$+ \tilde{f}_1(x)\exp(\tilde{\Gamma}(x))\ln\frac{\alpha^0 - x}{\alpha^0 + x} \tag{7.14}$$

Taking into account the fact that the functions $\tilde{f}_1(x)$ (7.2) and $\alpha(x)$ (7.9) have first and second derivatives in the interval $-\alpha^0 \leq x \leq \alpha^0$, the convergence of the parametric integrals on the right-hand sides of Eqs. (7.13) and (7.14) to continuous functions of x over the whole of the interval $-\alpha^0 \leq x \leq \alpha^0$ can be proved. On the basis of this assertion and using formulae (7.13) and (7.14), we find the singularities of the two versions of the solution $\varphi^{\pm}_{1h}(x)$ (7.8) of SIE (7.1) at the ends of the interval of integration when $x \to \pm\alpha^0$

$$\varphi^{+}_{1h}(x) \sim N^{+}_{1h}\ln(\alpha^0 \mp x), \quad \varphi^{-}_{1h}(x) \sim N^{-}_{1h}(\alpha^0 \mp x)^{-\theta} \tag{7.15}$$

The constants $N^{\pm}_{1h}$, when the expression for $\tilde{c}$ (7.12) is taken into account, are given by the formulae

$$N^{+}_{1h} = \frac{A_{wq12}\mu\alpha^0\tilde{f}_1(\alpha^0)}{\pi\Delta(\alpha^0)}, \quad N^{-}_{1h} = \tilde{c}\int_{-\alpha^0}^{\alpha^0} \frac{\tilde{f}_1(t)\exp(\tilde{\Gamma}(t)) - \tilde{f}_1(\alpha^0)\exp(\tilde{\Gamma}(\alpha^0))}{t-\alpha^0}dt \tag{7.16}$$

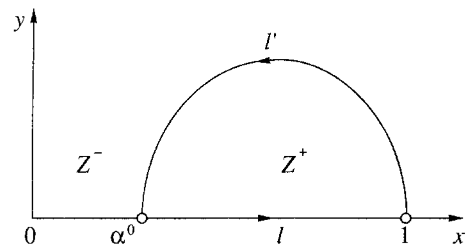

Fig. 2

We will now construct the closed analytical solution of the characteristic matrix system of SIE (5.39). Initially, we close the contour of integration of the SIE (5.39) $l = (\alpha^0 \le x \le 1)$ with a smooth arc l' and change to an equivalent SIE on the closed contour $L = l + l'$ (Fig. 2)

$$\tilde{A}\varphi(x) + \frac{\tilde{B}}{\pi i}\int_L \frac{\varphi(t)}{t-x}dt = A_{pq}\tilde{f}(x) \tag{7.17}$$

where the (4×4) square matrices $\tilde{A}$ and $\tilde{B}$ and the (4×1) column matrix $\tilde{f}(x)$ have the form

$$\tilde{A} = \begin{Vmatrix} A & O \\ O & E \end{Vmatrix}, \quad \tilde{B} = \begin{Vmatrix} B & O \\ O & O \end{Vmatrix}, \quad \tilde{f}(x) = \begin{Vmatrix} f(x) \\ 0 \end{Vmatrix} \tag{7.18}$$

where E is the identity matrix and O is a second-order zero matrix or a zero column matrix with two elements in the case of $\tilde{f}(x)$.

The closed analytical solution of SIE (7.17), (7.18) also holds for the initial SIE (5.39), and, using a Cauchy-type integral in the complex plane $z = x + iy$, we construct

$$\Phi(z) = \frac{1}{2\pi i}\int_L \frac{\varphi(t)}{t-z}dt \tag{7.19}$$

The contour $L = l + l'$ subdivides the z plane into two domains Z^+ and Z^-, which are respectively to the left and right of the direction of its circuit shown by the arrow in Fig. 2. The Sokhotskii–Plemelj formulae hold for the boundary values of the integral (7.19)

$$\Phi^{\pm}(x) = \pm\frac{1}{2}\varphi(x) + \frac{1}{2\pi i}\int_L \frac{\varphi(t)}{t-x}dt$$

from which the equalities

$$\varphi(x) = \Phi^+(x) - \Phi^-(x), \quad \frac{1}{\pi i}\int_L \frac{\varphi(t)}{t-x}dt = \Phi^+(x) + \Phi^-(x) \tag{7.20}$$

follow.

Substitution of expressions (7.20) into SIE (7.17) reduces it to a Riemann–Hilbert inhomogeneous conjugation problem

$$\Phi^+(x) = \tilde{G}\Phi^-(x) + \tilde{g}(x) \tag{7.21}$$

where

$$\begin{gathered} \tilde{G} = \tilde{S}^{-1}\tilde{D}, \quad \tilde{g}(x) = \tilde{S}^{-1}A_{pq}\tilde{f}(x) \\ \tilde{S} = \tilde{A} + \tilde{B} = \begin{Vmatrix} S & 0 \\ 0 & E \end{Vmatrix}, \quad \tilde{D} = \tilde{A} - \tilde{B} = \begin{Vmatrix} D & 0 \\ 0 & E \end{Vmatrix}, \quad S = A + B, \quad D = A - B \end{gathered} \tag{7.22}$$

Using formulae (5.40), (7.18) and (7.22) for the matrices A and B, $\tilde{A}$ and $\tilde{B}$ and $\tilde{S}$ and $\tilde{D}$ respectively, it is easy to verify that the equality $\det\tilde{G} = \det S^{-1}D = 1$ holds, from which it follows that the conjugation problem (7.21) has a zero index κ [3, 4]:

$$\kappa = \frac{1}{2\pi}[\arg\det\tilde{G}]_L = 0 \tag{7.23}$$

In the case of a constant matrix $\tilde{G}$, the homogeneous conjugation problem is solved in an explicit form using the canonical matrix

$$X(z) = \begin{cases} \tilde{G} \text{ when } z \in Z^+ \\ \tilde{E} \text{ when } z \in Z^- \end{cases}, \quad \tilde{E} = \begin{Vmatrix} E & 0 \\ 0 & E \end{Vmatrix} \tag{7.24}$$

Consequently, $\Phi^+ = X^+ = \tilde{G}$, $\Phi^- = X^- = \tilde{E}$ and, therefore

$$X^+ = \tilde{G}X^-, \quad \tilde{G} = X^+[X^-]^{-1} \tag{7.25}$$

The solution of the inhomogeneous conjugation problem (7.21), taking account of the zero index $\kappa = 0$ (7.23) is given by the following Cauchy-type integral [3, 4]

$$X^{-1}\Phi(z) = \frac{1}{2\pi i}\int_L \frac{[X^+]^{-1}\tilde{g}(t)}{t-z}dt \tag{7.26}$$

On applying the Sokhotskii–Plemelj boundary formulae to the integral (7.26), by analogy with equalities (7.20) we establish the relations

$$[X^+]^{-1}\tilde{g}(z) = [X^+]^{-1}\Phi^+(x) - [X^-]^{-1}\Phi^-(x)$$

$$\frac{1}{\pi i}\int_L \frac{[X^+]^{-1}\tilde{g}(t)}{t-x}dt = [X^+]^{-1}\Phi^+(x) + [X^-]^{-1}\Phi^-(x)$$

from which we find

$$\Phi^{\pm}(x) = X^{\pm}\left\{\pm\frac{1}{2}[X^+]^{-1}\tilde{g}(x) + \frac{1}{2\pi i}\int_L \frac{[X^+]^{-1}\tilde{g}(t)}{t-x}dt\right\} \tag{7.27}$$

Using the first formula of (7.20 and taking account of formulae (7.27) and the equalities $X^+ = \tilde{G} = \tilde{S}^{-}\tilde{D}, X^- = \tilde{E}, \tilde{g}(x) = A_{pq}\tilde{S}^{-1}\tilde{f}(x)$, we find the required function $\varphi(x)$ of the characteristic SIE (7.17)

$$\varphi(x) = \frac{A_{pq}}{2}\left(\tilde{S}^{-1}\tilde{f}(x) + \frac{\tilde{S}^{-1}}{\pi i}\int_L \frac{\tilde{f}(t)}{t-x}dt\right) + \frac{A_{pq}}{2}\left(\tilde{G}^{-1}\tilde{S}^{-1}\tilde{f}(x) - \frac{\tilde{G}^{-1}\tilde{S}^{-1}}{\pi i}\int_L \frac{\tilde{f}(t)}{t-x}dt\right) \tag{7.28}$$

Then, on taking account of the equality $\tilde{G}^{-1} = [\tilde{S}^{-1}\tilde{D}]^{-1} = \tilde{D}^{-1}\tilde{S}$, $\tilde{G}^{-1}\tilde{S}^{-1} = \tilde{D}^{-1}\tilde{S}\tilde{S}^{-1} = \tilde{D}^{-1}$ and the formulae

$$\tilde{S}^{-1} = \begin{Vmatrix} S^{-1} & 0 \\ 0 & E \end{Vmatrix}, \quad S^{-1} = \frac{1}{A_{pq}}\begin{Vmatrix} A_{uq12} & iA_{up12} \\ -iA_{wq12} & A_{wp12} \end{Vmatrix}$$

$$\tilde{D}^{-1} = \begin{Vmatrix} D^{-1} & 0 \\ 0 & E \end{Vmatrix}, \quad D^{-1} = \frac{1}{A_{pq}}\begin{Vmatrix} A_{uq12} & -iA_{up12} \\ iA_{wq12} & A_{wp12} \end{Vmatrix}$$

we represent $\varphi(x)$ (7.28) in the following final form

$$\varphi(x) = \tilde{A}_* \tilde{f}(x) + \frac{\tilde{B}_*}{\pi} \int_L \frac{\tilde{f}(t)}{t-x} dt \tag{7.29}$$

where

$$\tilde{A}_* = \frac{1}{2}(\tilde{S}^{-1} + \tilde{D}^{-1}) = \begin{Vmatrix} A_* & 0 \\ 0 & E \end{Vmatrix}, \quad A_* = \begin{Vmatrix} A_{uq12} & 0 \\ 0 & A_{wp12} \end{Vmatrix}$$
$$\tilde{B}_* = \frac{1}{2}(\tilde{S}^{-1} - D^{-1}) = \begin{Vmatrix} B_* & 0 \\ 0 & 0 \end{Vmatrix}, \quad B_* = \begin{Vmatrix} 0 & A_{up12} \\ -A_{wq12} & 0 \end{Vmatrix} \tag{7.30}$$

It is easily seen that the solution of the initial characteristic SIE (5.39) is identical to the solution of SIE (7.17) on the contour $l = (\alpha^0 \le x \le 1)$ and is given by formula (7.29):

$$\varphi(x) = A_* f(x) + \frac{B_*}{\pi} \int_{\alpha^0}^{1} \frac{f(t)}{t-x} dt \tag{7.31}$$

SIE (7.17) on the contour l' has the obvious trivial solution $\varphi(x) \equiv 0$ which is no longer required.
Next, taking account of formula (5.40) for $\varphi(x)$ and $f(x)$, we represent the matrix solution (7.31) for the characteristic SIE (5.39) in the expanded form

$$\varphi_{2h}(x) = A_{uq12} f_2(x) + \frac{A_{up12}}{\pi} \int_{\alpha^0}^{1} \frac{f_3(t)}{t-x} dt, \quad \varphi_{3h}(x) = A_{wp12} f_3(x) - \frac{A_{wq12}}{\pi} \int_{\alpha^0}^{1} \frac{f_2(t)}{t-x} dt \tag{7.32}$$

We substitute the representations of the functions $f_j(x)$ ($j = 2, 3$) into formulae (7.32), identically transform the integrals

$$\int_{\alpha^0}^{1} \frac{f_j^0(t)dt}{\sqrt{t-\alpha^0}(t-x)} = \tilde{f}_j^0(x) + f_j^0(x) \int_{\alpha^0}^{1} \frac{dt}{\sqrt{t-\alpha^0}(t-x)}, \quad \tilde{f}_j^0(x) = \int_{\alpha^0}^{1} \frac{f_j^0(t) - f_j^0(x)}{\sqrt{t-\alpha^0}(t-x)} dt \tag{7.33}$$

$$\int_{\alpha^0}^{1} \frac{dt}{\sqrt{t-\alpha^0}(t-x)} dt = \frac{1}{\sqrt{x-\alpha^0}} \ln \frac{1-x}{(\sqrt{1-\alpha^0} + \sqrt{x-\alpha^0})^2}$$

and separate from the singularities at the ends of the integration interval

$$\int_{\alpha^0}^{1} \frac{f_j^0(t)dt}{\sqrt{t-\alpha^0}(t-x)} = \frac{\ln(1-x)}{\sqrt{x-\alpha^0}} \left(\frac{\sqrt{x-\alpha^0}}{\ln(1-x)} \tilde{f}_j^0(x) + y(x) f_j^0(x) \right)$$
$$y(x) = 1 - 2\frac{\ln(\sqrt{1-\alpha^0} + \sqrt{x-\alpha^0})}{\ln(1-x)} \tag{7.34}$$

All of the functions $y(x), \tilde{f}_j^0(x), f_j^0(x)$ ($j = 2, 3$) are continuous in the interval $\alpha^0 \le x \le 1$.
The solution $\varphi_{jh}(x)$ ($j = 2, 3$) (7.23), taking account of $f_j(x)$ (5.40) and the quality (7.34), is written as follows with the singularities separated out in explicit form

$$\varphi_{jh}(x) = \frac{2}{\pi} \frac{\ln(1-x)}{\sqrt{x-\alpha^0}} \Phi_{jh}(x), \quad j = 2, 3, \quad \alpha^0 \le x \le 1$$

$$\Phi_{2h}(x) = A_{uq12}\frac{f_2^0(x)}{\ln(1-x)} + \frac{A_{up12}}{\pi}\left(y(x)f_3^0(x) + \frac{\sqrt{x-\alpha^0}}{\ln(1-x)}\tilde{f}_3^0(x)\right) \tag{7.35}$$

$$\Phi_{3h}(x) = A_{wp12}\frac{f_3^0(x)}{\ln(1-x)} - \frac{A_{wq12}}{\pi}\left(y(x)f_2^0(x) + \frac{\sqrt{x-\alpha^0}}{\ln(1-x)}\tilde{f}_2^0(x)\right)$$

The functions $\Phi_{jh}(x)$ ($j = 2, 3$) are continuous over the whole interval $\alpha^0 \le x \le 1$.

8. APPENDIX 2. REGULARIZATION OF THE SYSTEM OF SIE (5.29)–(5.31)

We will first regularize SIE (5.29) using the closed solution (7.8) of the characteristic SIE (7.1). We transfer all the regular terms to the right-hand side and apply formula (7.8) to them having replaced t by τ in this formula, as well as to the function $\tilde{f}_1(x)$. Next, we change the order of integration in the double integrals and transfer all the terms containing the unknown functions $\varphi_j(x)$ ($j = 1, 2, 3$) to the left-hand side. As a result, we obtain the following regularized equations for the two versions of the solution of SIE (5.29)

$$\varphi_1^{\pm}(x) + \frac{A_{pq}}{\pi}\int_{-\alpha^0}^{\alpha^0} M_{11}^{\pm}(x,t)\varphi_1^{\pm}(t)dt + \frac{2A_{pq}}{\pi}\sum_{j=2}^{3}\int_{\alpha^0}^{1}\frac{M_{1j}^{\pm}(x,t)}{\sqrt{t-\alpha^0}}\varphi_j^{\pm}(t)dt = \varphi_{1h}^{\pm}(x) \tag{8.1}$$

$$M_{1j}^{\pm} = \frac{A_{wp12}K_{1j}(x,t)}{\Delta(x)} + \frac{A_{wq12}\mu x}{\pi\Delta(x)}\exp(\pm\tilde{\Gamma}(x))\int_{-\alpha^0}^{\alpha^0}\frac{K_{1j}(\tau,t)\exp(\mp\tilde{\Gamma}(\tau))}{\tau-x}d\tau, \quad j = 1, 2, 3 \tag{8.2}$$

It is easy to verify that the functions $\varphi_{1h}^{\mp}(x)$ and the kernels $M_{1j}^{\mp}(x, t)$ ($j = 1, 2, 3$) (8.2) are even in the variable x and that the kernels $M_{11}^{\mp}(x, t)$ are also even in the variable t. In this case, the required functions $\varphi_1^{\mp}(x)$ in the two versions of Eq. (8.1) are even, and, therefore, it is possible and advisable to represent them in the following form

$$\varphi_1^{\pm}(x) + \frac{2A_{pq}}{\pi}\left(\int_0^{\alpha^0} M_{11}^{\pm}(x,t)\varphi_1^{\pm}(t)dt + \sum_{j=2}^{3}\int_{\alpha^0}^{1}\frac{M_{1j}^{\pm}(x,t)}{\sqrt{t-\alpha^0}}\varphi_j^{\pm}(t)dt\right) = \varphi_{1h}^{\pm}(x) \tag{8.3}$$

Comparing the expressions $\varphi_{1h}^{\mp}(x)$ (7.8) and $M_{1j}^{\mp}(x, t)$ (8.2) and taking account of the differentiability of the functions $K_{1j}(x, t)$ ($j = 1, 2, 3$) with respect to the variable x, it can be seen that the kernels $M_{1j}^{\mp}(x, t)$ ($j = 1, 2, 3$) and the free terms $\varphi_{1h}^{\mp}(x)$ of Eqs (8.3) have the same singularities (7.15) at the end $x = \alpha^0$.

In fact, by representing the integrals on the right-hand side of equality (8.2) using formulae (7.13) and (7.14) and, at the same time, replacing $\tilde{f}_1(x)$ by $K_{1j}(x, t)$, we find the leading terms of the kernels $M_{1j}^{\mp}(x, t)$ ($j = 1, 2, 3$) when $x \to \alpha^0$ using the technique described above

$$M_{1j}^{+}(x,t) \sim N_{1j}^{+}(t)\ln(\alpha^0 - x), \quad M_{1j}^{-}(x,t) \sim N_{1j}^{-}(t)(\alpha^0 - x)^{-\theta} \tag{8.4}$$

where the functions $N_{1j}^{\mp}(t)$, when account is taken of $\tilde{c}$ (7.12), are given by the following formula

$$N_{1j}^{+}(t) = \frac{A_{wq12}\mu\alpha^0 K_{1j}(\alpha^0,t)}{\pi\Delta(\alpha^0)}$$

$$N_{1j}^{-}(t) = \tilde{c}\int_{-\alpha^0}^{\alpha^0}\frac{K_{1j}(\tau,t)\exp(\tilde{\Gamma}(\tau)) - K_{1j}(\alpha^0,t)\exp(\tilde{\Gamma}(\alpha^0))}{\tau-\alpha^0}d\tau \tag{8.5}$$

In order to remove the logarithmic singularities (7.15) and (8.4) in Eq. (8.3) for the functions $\varphi_j^+(x)$, ($j = 1, 2, 3$), we introduce the new unknown functions

$$\Phi_1^+(x) = \frac{\varphi_1^+(x)}{\ln(\alpha^0 - x)}, \quad \Phi_j^+(x) = \frac{\varphi_j^+(x)}{\ln(1-x)}, \quad j = 2, 3 \tag{8.6}$$

and obtain for them the integral equation

$$\Phi_1^+(x) + \frac{2A_{pq}}{\pi}\left(\int_0^{\alpha^0} \tilde{M}_{11}^+(x,t)\ln(\alpha^0 - t)\Phi_1^+(t)dt + \right.$$
$$\left. + \sum_{j=2}^{3}\int_{\alpha^0}^{1} \frac{\tilde{M}_{1j}^+(x,t)\ln(1-t)}{\sqrt{t-\alpha^0}}\Phi_j^+(t)dt\right) = \Phi_{1h}^{\pm}(x), \quad 0 \le x \le \alpha^0 \tag{8.7}$$

where $\tilde{M}_{1j}^+(x, t) = M_{1j}^+(x, t)/\ln(\alpha_0 - t)$ ($j = 1, 2, 3$), $\Phi_{1h}^+(x) = \varphi_{1h}^+(x)/\ln(\alpha_0 - t)$ are continuous functions in the corresponding intervals of integration and, when $x = \alpha^0$, we have

$$\tilde{M}_{1j}^+(\alpha^0, t) = N_{1j}^+(t), \quad j = 1, 2, 3, \quad \Phi_{1h}^+(\alpha^0) = N_{1h}^+ \tag{8.8}$$

We remove the root singularity in the second and third integrals of Eq. (8.7) by changing the variable of integration using the formulae

$$t_2(t) = \sqrt{t - \alpha^0}, \quad t(t_2) = \alpha^0 + t_2^2 \tag{8.9}$$

As a result, we arrive at the integral equation

$$\Phi_1^+(x) + \frac{2A_{pq}}{\pi}\left(\int_0^{\alpha^0} \tilde{M}_{11}^+(x,t)\ln(\alpha^0 - t)\Phi_1^+(t)dt + \right.$$
$$\left. + 2\sum_{j=2}^{3}\int_0^{\sqrt{1-\alpha^0}} \tilde{M}_{1j*}^+(x,t)\ln(1 - t(t_2))\Phi_{j*}^+(t_2)dt_2\right) = \Phi_{1h}^+(x), \quad 0 \le x \le \alpha^0 \tag{8.10}$$

where the functions

$$\tilde{M}_{1j*}^+(x,t) = \tilde{M}_{1j}^+(x, t(t_2)), \quad j = 1, 2, 3, \quad \Phi_{j*}^+(t_2) = \Phi_j^+(t(t_2)), \quad j = 2, 3$$

are labelled with an asterisk.

We remove the logarithmic singularities in the integrals of Eq. (8.10) by representing them in the following forms

$$\int_0^{\alpha^0} \tilde{M}_{11}^+(x,t)\ln(\alpha^0 - t)\Phi_1^+(t)dt = \int_0^{\alpha^0} \tilde{M}_{11}^+(x,t)\ln(\alpha^0 - t)(\Phi_1^+(t) - \Phi_1^+(\alpha^0))dt +$$
$$+ \Phi_1^+(\alpha^0)\left(\int_0^{\alpha^0} (\tilde{M}_{11}^+(x,t) - \tilde{M}_{11}^+(x,\alpha^0))\ln(\alpha^0 - t)dt + \tilde{M}_{11}^+(x,\alpha^0)\int_0^{\alpha^0} \ln(\alpha^0 - t)dt\right) \tag{8.11}$$

$$\int_0^{\alpha^0} \ln(\alpha^0 - t)dt = -\alpha^0(1 - \ln\alpha^0)$$

$$\int_0^{\sqrt{1-\alpha^0}} \tilde{M}_{1j*}^{+}(x, t_2)\ln(1-t(t_2))\Phi_{j*}^{+}(t_2)dt_2 =$$

$$= \int_0^{\sqrt{1-\alpha^0}} \tilde{M}_{1j*}^{+}(x, t_2)\ln(1-t(t_2))(\Phi_{j*}^{+}(t_2) - \Phi_{j*}^{+}(\sqrt{1-\alpha^0}))dt +$$

$$+\Phi_{j*}^{+}(\sqrt{1-\alpha^0})\left(\int_0^{\sqrt{1-\alpha^0}} (\tilde{M}_{1j*}^{+}(x, t_2) - \tilde{M}_{1j*}^{+}(x, \sqrt{1-\alpha^0}))\ln(1-t(t_2))dt_2 + \right. \tag{8.12}$$

$$\left. + \tilde{M}_{1j*}^{+}(x, \sqrt{1-\alpha^0}) \int_0^{\sqrt{1-\alpha^0}} \ln(1-t(t_2))dt_2 \right)$$

$$\int_0^{\sqrt{1-\alpha^0}} \ln(1-t(t_2))dt_2 = \int_0^{\sqrt{1-\alpha^0}} \ln(1-\alpha^0-t_2^2)dt_2 = -2\sqrt{1-\alpha^0}(1-\ln(2\sqrt{1-\alpha^0}))$$

To remove the power singularity (7.15) and the logarithmic singularity (7.35) in Eq. (8.3) for the functions $\varphi_j^-(x)$ ($j = 1, 2, 3$), we introduce the new unknown functions

$$\Phi_1^-(x) = (\alpha^0 - x)^{\theta}\varphi_1^-(x), \quad \Phi_j^-(x) = \frac{\varphi_j^-(x)}{\ln(1-x)}, \quad j = 2, 3 \tag{8.13}$$

and obtain the following integral equation for them

$$\Phi_1^-(x) + \frac{2A_{pq}}{\pi}\left(\int_0^{\alpha^0} \frac{\tilde{M}_{11}^-(x, t)}{(\alpha^0 - t)^{\theta}}\Phi_1^-(t)dt + \sum_{j=2}^{3} \int_{\alpha^0}^{1} \frac{\tilde{M}_{1j}^-(x, t)\ln(1-t)}{\sqrt{t-\alpha^0}}\Phi_j^-(t)dt\right) = \Phi_{1h}^-(x) \tag{8.14}$$

$$0 \le x \le \alpha^0$$

For the continuous functions

$$\tilde{M}_{1j}^-(x, t) = M_{1j}^-(x, t)(\alpha^0 - x)^{\theta}, \quad j = 1, 2, 3, \quad \Phi_{1h}^-(x) = \varphi_{1h}^-(x)(\alpha^0 - x)^{\theta} \tag{8.15}$$

in the corresponding domains in which they are defined when $x = \alpha^0$, we have

$$\tilde{M}_{1j}^-(\alpha^0, t) = N_{1j}^-(t), \quad j = 1, 2, 3, \quad \Phi_{1h}^-(\alpha^0) = N_{1h}^- \tag{8.16}$$

We remove the root singularity in the second and third integrals of Eq. (8.14) by changing the variable of integration t using formulae (8.9), while the power singularity in the first integral is eliminated by changing the variable of integration t using the formulae

$$t_1(t) = \int_0^{t} \frac{dt}{(\alpha^0 - t)^{\theta}} = \frac{\alpha^{01-\theta} - (\alpha^0 - t)^{1-\theta}}{1-\theta}, \quad t(t_1) = \alpha^0 - (\alpha^{01-\theta} - (1-\theta)t_1)^{1/(1-\theta)} \tag{8.17}$$

and, using the same formulae, we replace the variable x by x_1. As a result, the integral equation becomes

$$\Phi_{1*}^{-}(x_1)+\frac{2A_{pq}}{\pi}\left(\int_0^{t_1(\alpha^0)}\tilde{M}_{11*}^{-}(x_1,t_1)\Phi_{1*}^{-}(t_1)dt_1+\right.$$

$$\left.+2\sum_{j=2}^{3}\int_0^{\sqrt{1-\alpha^0}}\tilde{M}_{1j*}^{-}(x_1,t_2)\ln(1-t(t_2))\Phi_{j*}^{-}(t_2)dt_2\right)=\Phi_{1h*}^{-}(x_1),\quad 0\le x_1\le x_1(\alpha^0) \tag{8.18}$$

where

$$t_1(\alpha^0)=\frac{\alpha^{01-\theta}}{1-\theta},\quad \Phi_{1*}^{-}(x_1)=\Phi_1^{-}(x(x_1)),\quad \Phi_{j*}^{-}(x_2)=\Phi_j^{-}(x(x_2)),\quad j=2,3$$

$$\tilde{M}_{11*}^{-}(x_1,t_1)=\tilde{M}_{11}^{-}(x(x_1),t(t_1)),\quad \tilde{M}_{1j*}^{-}(x_1,t_2)=\tilde{M}_{1j}^{-}(x(x_1),t(t_2)),\ j=2,3$$

$$\Phi_{1h*}^{-}(x_1)=\Phi_{1h}^{-}(x(x_1))$$

The second and third integrals in Eq. (8.18) are represented in the form of (8.12) with x replaced by $x(x_1)$.

We now regularize SIE (5.30) and (5.31) and, using the closed solution (7.32), the characteristic system of SIE (5.40). We transfer all the regular terms of Eqs (5.30) and (5.31) to the right-hand side and apply formulae (7.32) (having replaced t by τ in them) to the equations and the functions $\bar{f}_j(x)$ ($j=1,2,3$). We then change the order of integration in the double integrals and transfer all the terms containing the unknown functions $\varphi_j(x)$ ($j=1,2,3$) to the left-hand side. As a result, we obtain the following regular equations

$$\varphi_j(x)+\frac{2}{\pi}\int_0^{\alpha^0}\frac{M_{j1}(x,t)}{\sqrt{x-\alpha^0}}\varphi_1(t)dt+\frac{2}{\pi}\sum_{n=2}^{3}\int_{\alpha^0}^{1}\frac{M_{jn}(x,t)\varphi_n(t)}{\sqrt{(x-\alpha^0)(t-\alpha^0)}}dt=\varphi_{jh}(x),\quad \alpha^0\le x\le 1 \tag{8.19}$$

where the kernels $M_{jn}(x,t)$ ($j=2,3$; $n=1,2,3$) are given by the formulae

$$\begin{aligned}M_{2n}(x,t)&=A_{uq12}K_{2n}(x,t)+\frac{A_{up12}}{\pi}\sqrt{x-\alpha^0}\int_{\alpha^0}^{1}\frac{K_{3n}(\tau,t)}{\sqrt{(\tau-\alpha^0)(\tau-x)}}d\tau\\ M_{3n}(x,t)&=A_{wp12}K_{3n}(x,t)-\frac{A_{wq12}}{\pi}\sqrt{x-\alpha^0}\int_{\alpha^0}^{1}\frac{K_{2n}(\tau,t)}{\sqrt{(\tau-\alpha^0)(\tau-x)}}d\tau\end{aligned} \tag{8.20}$$

Using equality (7.34), we separate out the singularities at the ends of the interval of integration from the integrals in formulae (8.20) and represent them in the form

$$\begin{aligned}&\int_{\alpha^0}^{1}\frac{K_{jn}(\tau,t)}{\sqrt{(\tau-\alpha^0)(\tau-x)}}d\tau=\frac{\ln(1-x)}{\sqrt{x-\alpha^0}}\left(\frac{\sqrt{x-\alpha^0}}{\ln(1-x)}\tilde{K}_{jn}(x,t)+y(x)K_{jn}(x,t)\right)\\ &\tilde{K}_{jn}(x,t)=\int_{\alpha^0}^{1}\frac{K_{jn}(\tau,t)-K_{jn}(x,t)}{\sqrt{(\tau-\alpha^0)(\tau-x)}}d\tau,\quad j=2,3;\quad n=1,2,3\end{aligned} \tag{8.21}$$

We now introduce the representations of the free terms $\varphi_{jh}(x)$ (7.35) and of the integrals (8.21) into Eqs (8.19), multiply them by $\sqrt{x-\alpha^0}$ and divide them by $\ln(1-x)$, and introduce the new unknown functions for both versions of the solution

$$\Phi_1^{+}(x)=\frac{\varphi_1^{+}(x)}{\ln(\alpha^0-x)},\quad \Phi_1^{-}(x)=(\alpha^0-x)^{\theta}\varphi_1^{-}(x),\quad \Phi_j^{\pm}(x)=\frac{\varphi_j^{\pm}(x)}{\ln(1-x)}\ (j=2,3)$$

As a result, we obtain the following systems of integral equations for them

$$\frac{2}{\pi}\sqrt{x-\alpha^0}\Phi_j^+(x)+\int_0^{\alpha^0}\tilde{M}_{j1}^+(x,t)\ln(\alpha^0-t)\Phi_1^+(t)dt+$$
$$+\sum_{m=2}^{3}\int_{\alpha^0}^{1}\frac{\tilde{M}_{jn}^+(x,t)\ln(1-t)}{\sqrt{t-\alpha^0}}\Phi_j^+(t)dt = \Phi_{jh}(x),\quad j=2,3;\quad \alpha^0\le x\le 1 \tag{8.22}$$

$$\frac{2}{\pi}\sqrt{x-\alpha^0}\Phi_j^-(x)+\int_0^{\alpha^0}\frac{\tilde{M}_{j1}^-(x,t)}{(\alpha^0-t)^{\theta}}\Phi_1^-(t)dt+$$
$$+\sum_{n=2}^{3}\int_{\alpha^0}^{1}\frac{\tilde{M}_{jn}^-(x,t)\ln(1-t)}{\sqrt{t-\alpha^0}}\Phi_j^-(t)dt = \Phi_{jh}(x),\quad j=2,3;\quad \alpha^0\le x\le 1 \tag{8.23}$$

where the kernels $\tilde{M}_{jn}^{\pm}(x,t)$ $(j = 2, 3;\ n = 1, 2, 3)$ are given by the formulae

$$\begin{aligned}\tilde{M}_{2n}^{\pm}(x,t) &= A_{uq12}\frac{K_{2n}(x,t)}{\ln(1-x)}+\frac{A_{up12}}{\pi}\left(\frac{\sqrt{x-\alpha^0}}{\ln(1-x)}\tilde{K}_{2n}(x,t)+y(x)K_{2_n}(x,t)\right)\\ \tilde{M}_{3n}^{\pm}(x,t) &= A_{wp12}\frac{K_{3n}(x,t)}{\ln(1-x)}-\frac{A_{wq12}}{\pi}\left(\frac{\sqrt{t-\alpha^0}}{\ln(1-x)}\tilde{K}_{3n}(x,t)+y(x)K_{3_n}(x,t)\right)\end{aligned} \tag{8.24}$$

and, when $x = 1$, we have

$$\tilde{M}_{2n}^{\pm}(1,t) = \frac{A_{up12}}{\pi}K_{2_n}(1,t),\quad \tilde{M}_{3n}^{\pm}(1,t) = -\frac{A_{wq12}}{\pi}K_{3_n}(1,t) \tag{8.25}$$

In order to remove the root singularities in the two systems of equations (8.22) and (8.23), we replace the independent variable x and the variable of integration t in the interval $[\alpha^0, 1]$ using formulae (8.9). In the first integral of system (8.23), we change to the variable of integration (8.17) and thereby remove the power singularity. As a result, the systems of equations (8.22) and (8.23) become

$$\frac{2}{\pi}x_2\Phi_{j*}^+(x_2)+\int_0^{\alpha^0}\tilde{M}_{j1*}^+(x_2,t)\ln(\alpha^0-t)\Phi_1^+(t)dt+$$
$$+2\sum_{n=2}^{3}\int_0^{\sqrt{1-\alpha^0}}\tilde{M}_{jn*}^+(x_2,t_2)\ln(1-t(t_2))\Phi_{j*}^+(t_2)dt_2 = \Phi_{jh*}(x_2),\quad 0\le x_2\le\sqrt{1-\alpha^0} \tag{8.26}$$

$$\frac{2}{\pi}x_2\Phi_{j*}^-(x_2)+\int_0^{t_1(\alpha^0)}\tilde{M}_{j1*}^-(x_2,t_1)\Phi_{1*}^-(t_1)dt_1+$$
$$+2\sum_{n=2}^{3}\int_0^{\sqrt{1-\alpha^0}}\tilde{M}_{jn*}^-(x_2,t_2)\ln(1-t(t_2))\Phi_{j*}^-(t_2)dt_2 = \Phi_{jh*}(x_2),\quad 0\le x_2\le\sqrt{1-\alpha^0} \tag{8.27}$$

The functions $\tilde{M}_{j1*}^{\pm}$, $\Phi_{j*}^{\pm}$ and Φ_{jh*} are related to the functions $\tilde{M}_{j1}^{\pm}$, $\Phi_j^{\pm}$ and Φ_{jh} using formulae (8.18).
The logarithmic singularities at the ends of the intervals of integration $x = \alpha^0$ and $x = 1$ in the system of equations ((8.26), (8.27) are removed by representing the integrals using formulae (8.11) and (8.12).
Thus, as a result of regularization, the system of SIE is reduced to two systems of three linear Fredholm integral equations (8.10), (8.26) and (8.18), (8.27) of the second and third kinds with continuous kernels

and free terms, which solve the initial problem of dynamics over the whole range of changes in the elastic and velocity characteristics of the crack propagation respectively in the case of positive and negative signs of the ratio $\xi = A_{wq12}/A_{wp12}$. By Fredholm's theorem, these systems have unique bounded solutions, which can be obtained by standard computational methods.

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